

## Chapter 2

# Phase Plane Analysis

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Phase plane analysis is a graphical method for studying second-order systems, which was introduced well before the turn of the century by mathematicians such as Henri Poincaré. The basic idea of the method is to generate, in the state space of a second-order dynamic system (a two-dimensional plane called the phase plane), motion trajectories corresponding to various initial conditions, and then to examine the qualitative features of the trajectories. In such a way, information concerning stability and other motion patterns of the system can be obtained. In this chapter, our objective is to gain familiarity with nonlinear systems through this simple graphical method.

Phase plane analysis has a number of useful properties. First, as a graphical method, it allows us to visualize what goes on in a nonlinear system starting from various initial conditions, without having to solve the nonlinear equations analytically. Second, it is not restricted to small or smooth nonlinearities, but applies equally well to strong nonlinearities and to "hard" nonlinearities. Finally, some practical control systems can indeed be adequately approximated as second-order systems, and the phase plane method can be used easily for their analysis. Conversely, of course, the fundamental disadvantage of the method is that it is restricted to second-order (or first-order) systems, because the graphical study of higher-order systems is computationally and geometrically complex.

## 2.1 Concepts of Phase Plane Analysis

### 2.1.1 Phase Portraits

The phase plane method is concerned with the graphical study of second-order autonomous systems described by

$$\dot{x}_1 = f_1(x_1, x_2) \quad (2.1a)$$

$$\dot{x}_2 = f_2(x_1, x_2) \quad (2.1b)$$

where  $x_1$  and  $x_2$  are the states of the system, and  $f_1$  and  $f_2$  are nonlinear functions of the states. Geometrically, the state space of this system is a plane having  $x_1$  and  $x_2$  as coordinates. We will call this plane the *phase plane*.

Given a set of initial conditions  $\mathbf{x}(0) = \mathbf{x}_0$ , Equation (2.1) defines a solution  $\mathbf{x}(t)$ . With time  $t$  varied from zero to infinity, the solution  $\mathbf{x}(t)$  can be represented geometrically as a curve in the phase plane. Such a curve is called a *phase plane trajectory*. A family of phase plane trajectories corresponding to various initial conditions is called a *phase portrait* of a system.

To illustrate the concept of phase portrait, let us consider the following simple system.

#### Example 2.1: Phase portrait of a mass-spring system

The governing equation of the mass-spring system in Figure 2.1(a) is the familiar linear second-order differential equation

$$\ddot{x} + x = 0 \quad (2.2)$$

Assume that the mass is initially at rest, at length  $x_0$ . Then the solution of the equation is

$$x(t) = x_0 \cos t$$

$$\dot{x}(t) = -x_0 \sin t$$

Eliminating time  $t$  from the above equations, we obtain the equation of the trajectories

$$x^2 + \dot{x}^2 = x_0^2$$

This represents a circle in the phase plane. Corresponding to different initial conditions, circles of different radii can be obtained. Plotting these circles on the phase plane, we obtain a phase portrait for the mass-spring system (Figure 2.1.b).  $\square$

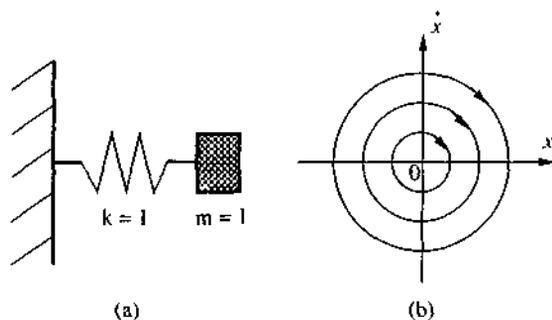


Figure 2.1 : A mass-spring system and its phase portrait

The power of the phase portrait lies in the fact that once the phase portrait of a system is obtained, the nature of the system response corresponding to various initial conditions is directly displayed on the phase plane. In the above example, we easily see that the system trajectories neither converge to the origin nor diverge to infinity. They simply circle around the origin, indicating the marginal nature of the system's stability.

A major class of second-order systems can be described by differential equations of the form

$$\ddot{x} + f(x, \dot{x}) = 0 \quad (2.3)$$

In state space form, this dynamics can be represented as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -f(x_1, x_2)$$

with  $x_1 = x$  and  $x_2 = \dot{x}$ . Most second-order systems in practice, such as mass-damper-spring systems in mechanics, or resistor-coil-capacitor systems in electrical engineering, can be represented in or transformed into this form. For these systems, the states are  $x$  and its derivative  $\dot{x}$ . Traditionally, the phase plane method is developed for the dynamics (2.3), and the phase plane is defined as the plane having  $x$  and  $\dot{x}$  as coordinates. But it causes no difficulty to extend the method to more general dynamics of the form (2.1), with the  $(x_1, x_2)$  plane as the phase plane, as we do in this chapter.

## 2.1.2 Singular Points

An important concept in phase plane analysis is that of a singular point. A singular point is an equilibrium point in the phase plane. Since an equilibrium point is defined as a point where the system states can stay forever, this implies that  $\dot{\mathbf{x}} = \mathbf{0}$ , and using (2.1),

$$f_1(x_1, x_2) = 0 \quad f_2(x_1, x_2) = 0 \quad (2.4)$$

The values of the equilibrium states can be solved from (2.4).

For a linear system, there is usually only one singular point (although in some cases there can be a *continuous* set of singular points, as in the system  $\ddot{x} + \dot{x} = 0$ , for which all points on the real axis are singular points). However, a nonlinear system often has more than one isolated singular point, as the following example shows.

### Example 2.2: A nonlinear second-order system

Consider the system

$$\ddot{x} + 0.6 \dot{x} + 3x + x^2 = 0$$

whose phase portrait is plotted in Figure 2.2. The system has two singular points, one at  $(0, 0)$  and the other at  $(-3, 0)$ . The motion patterns of the system trajectories in the vicinity of the two singular points have different natures. The trajectories move towards the point  $x = 0$  while moving away from the point  $x = -3$ .  $\square$

One may wonder why an equilibrium point of a second-order system is called a *singular point*. To answer this, let us examine the slope of the phase trajectories. From (2.1), the slope of the phase trajectory passing through a point  $(x_1, x_2)$  is determined by

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \quad (2.5)$$

With the functions  $f_1$  and  $f_2$  assumed to be single valued, there is usually a definite value for this slope at any given point in phase plane. This implies that the phase trajectories will not intersect. At singular points, however, the value of the slope is  $0/0$ , i.e., the slope is indeterminate. Many trajectories may intersect at such points, as seen from Figure 2.2. This indeterminacy of the slope accounts for the adjective "singular".

Singular points are very important features in the phase plane. Examination of the singular points can reveal a great deal of information about the properties of a

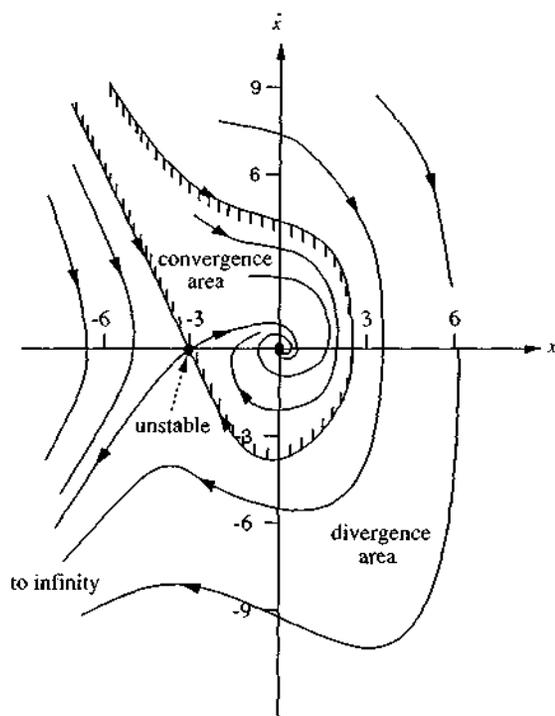


Figure 2.2 : The phase portrait of a nonlinear system

system. In fact, the stability of linear systems is uniquely characterized by the nature of their singular points. For nonlinear systems, besides singular points, there may be more complex features, such as limit cycles. These issues will be discussed in detail in sections 2.3 and 2.4.

Note that, although the phase plane method is developed primarily for second-order systems, it can also be applied to the analysis of first-order systems of the form

$$\dot{x} + f(x) = 0$$

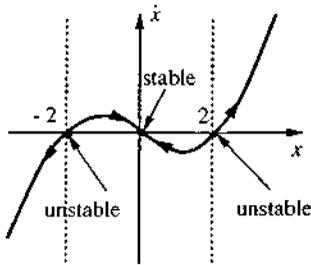
The idea is still to plot  $\dot{x}$  with respect to  $x$  in the phase plane. The difference now is that the phase portrait is composed of a single trajectory.

**Example 2.3: A first-order system**

Consider the system

$$\dot{x} = -4x + x^3$$

There are three singular points, defined by  $-4x + x^3 = 0$ , namely,  $x = 0, -2$ , and  $2$ . The phase-portrait of the system consists of a single trajectory, and is shown in Figure 2.3. The arrows in the figure denote the direction of motion, and whether they point toward the left or the right at a particular point is determined by the sign of  $\dot{x}$  at that point. It is seen from the phase portrait of this system that the equilibrium point  $x = 0$  is stable, while the other two are unstable.  $\square$



**Figure 2.3 :** Phase trajectory of a first-order system

### 2.1.3 Symmetry in Phase Plane Portraits

A phase portrait may have *a priori* known symmetry properties, which can simplify its generation and study. If a phase portrait is symmetric with respect to the  $x_1$  or the  $x_2$  axis, one only needs in practice to study half of it. If a phase portrait is symmetric with respect to both the  $x_1$  and  $x_2$  axes, only one quarter of it has to be explicitly considered.

Before generating a phase portrait itself, we can determine its symmetry properties by examining the system equations. Let us consider the second-order dynamics (2.3). The slope of trajectories in the phase plane is of the form

$$\frac{dx_2}{dx_1} = -\frac{f(x_1, x_2)}{\dot{x}}$$

Since symmetry of the phase portraits also implies symmetry of the slopes (equal in absolute value but opposite in sign), we can identify the following situations:

**Symmetry about the  $x_1$  axis:** The condition is

$$f(x_1, x_2) = f(x_1, -x_2)$$

This implies that the function  $f$  should be even in  $x_2$ . The mass-spring system in Example 2.1 satisfies this condition. Its phase portrait is seen to be symmetric about the  $x_1$  axis.

**Symmetry about the  $x_2$  axis:** Similarly,

$$f(x_1, x_2) = -f(-x_1, x_2)$$

implies symmetry with respect to the  $x_2$  axis. The mass-spring system also satisfies this condition.

**Symmetry about the origin:** When

$$f(x_1, x_2) = -f(-x_1, -x_2)$$

the phase portrait of the system is symmetric about the origin.

## 2.2 Constructing Phase Portraits

Today, phase portraits are routinely computer-generated. In fact, it is largely the advent of the computer in the early 1960's, and the associated ease of quickly generating phase portraits, which spurred many advances in the study of complex nonlinear dynamic behaviors such as chaos. However, of course (as *e.g.*, in the case of root locus for linear systems), it is still practically useful to learn how to roughly sketch phase portraits or quickly verify the plausibility of computer outputs.

There are a number of methods for constructing phase plane trajectories for linear or nonlinear systems, such as the so-called analytical method, the method of isoclines, the delta method, Lienard's method, and Pell's method. We shall discuss two of them in this section, namely, the analytical method and the method of isoclines. These methods are chosen primarily because of their relative simplicity. The analytical method involves the analytical solution of the differential equations describing the systems. It is useful for some special nonlinear systems, particularly piece-wise linear systems, whose phase portraits can be constructed by piecing together the phase portraits of the related linear systems. The method of isoclines is a graphical method which can conveniently be applied to construct phase portraits for systems which cannot be solved analytically, which represent by far the most common case.

## ANALYTICAL METHOD

There are two techniques for generating phase plane portraits analytically. Both techniques lead to a functional relation between the two phase variables  $x_1$  and  $x_2$  in the form

$$g(x_1, x_2, c) = 0 \quad (2.6)$$

where the constant  $c$  represents the effects of initial conditions (and, possibly, of external input signals). Plotting this relation in the phase plane for different initial conditions yields a phase portrait.

The first technique involves solving equations (2.1) for  $x_1$  and  $x_2$  as functions of time  $t$ , *i.e.*,

$$x_1(t) = g_1(t) \quad x_2(t) = g_2(t)$$

and then eliminating time  $t$  from these equations, leading to a functional relation in the form of (2.6). This technique was already illustrated in Example 2.1.

The second technique, on the other hand, involves directly eliminating the time variable, by noting that

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

and then solving this equation for a functional relation between  $x_1$  and  $x_2$ . Let us use this technique to solve the mass-spring equation again.

**Example 2.4: Mass-spring system**

By noting that  $\ddot{x} = (d\dot{x}/dx)(dx/dt)$ , we can rewrite (2.2) as

$$\dot{x} \frac{d\dot{x}}{dx} + x = 0$$

Integration of this equation yields

$$\dot{x}^2 + x^2 = x_0^2 \quad \square$$

One sees that the second technique is more straightforward in generating the equations for the phase plane trajectories.

Most nonlinear systems cannot be easily solved by either of the above two techniques. However, for piece-wise linear systems, an important class of nonlinear systems, this method can be conveniently used, as the following example shows.

## Example 2.5: A satellite control system

Figure 2.4 shows the control system for a simple satellite model. The satellite, depicted in Figure 2.5(a), is simply a rotational unit inertia controlled by a pair of thrusters, which can provide either a positive constant torque  $U$  (positive firing) or a negative torque  $-U$  (negative firing). The purpose of the control system is to maintain the satellite antenna at a zero angle by appropriately firing the thrusters. The mathematical model of the satellite is

$$\ddot{\theta} = u$$

where  $u$  is the torque provided by the thrusters and  $\theta$  is the satellite angle.

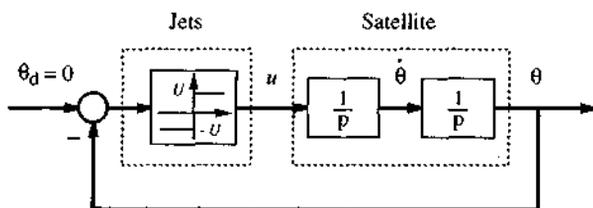


Figure 2.4 : Satellite control system

Let us examine on the phase plane the behavior of the control system when the thrusters are fired according to the control law

$$u(t) = \begin{cases} -U & \text{if } \theta > 0 \\ U & \text{if } \theta < 0 \end{cases} \quad (2.7)$$

which means that the thrusters push in the counterclockwise direction if  $\theta$  is positive, and vice versa.

As the first step of the phase portrait generation, let us consider the phase portrait when the thrusters provide a positive torque  $U$ . The dynamics of the system is

$$\ddot{\theta} = U$$

which implies that  $\dot{\theta} d\dot{\theta} = U d\theta$ . Therefore, the phase trajectories are a family of parabolas defined by

$$\dot{\theta}^2 = 2U\theta + c_1$$

where  $c_1$  is a constant. The corresponding phase portrait of the system is shown in Figure 2.5(b).

When the thrusters provide a negative torque  $-U$ , the phase trajectories are similarly found to be

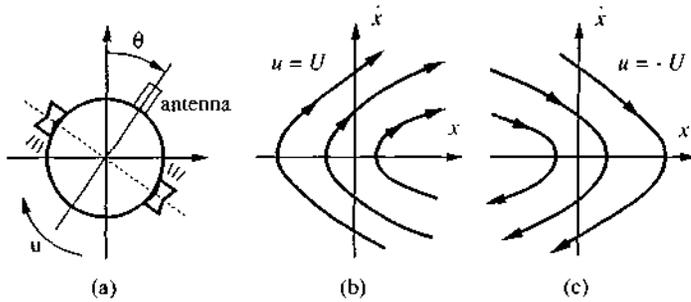


Figure 2.5 : Satellite control using on-off thrusters

$$\dot{\theta}^2 = -2Ux + c_1$$

with the corresponding phase portrait shown in Figure 2.5(c).

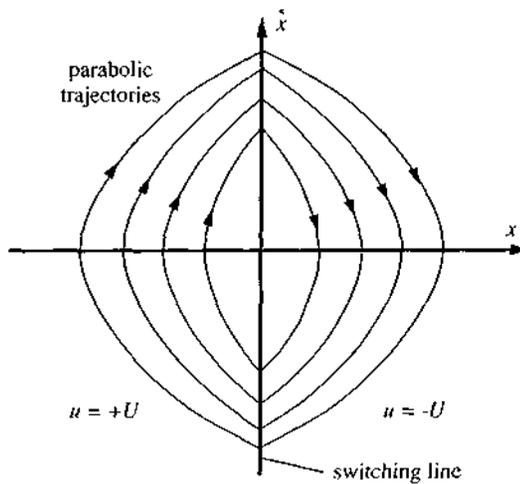


Figure 2.6 : Complete phase portrait of the control system

The complete phase portrait of the closed-loop control system can be obtained simply by connecting the trajectories on the left half of the phase plane in 2.5(b) with those on the right half of the phase plane in 2.5(c), as shown in Figure 2.6. The vertical axis represents a switching line, because the control input and thus the phase trajectories are switched on that line. It is interesting to see that, starting from a nonzero initial angle, the satellite will oscillate in periodic motions

under the action of the jets. One concludes from this phase portrait that the system is marginally stable, similarly to the mass-spring system in Example 2.1. Convergence of the system to the zero angle can be obtained by adding rate feedback (Exercise 2.4).  $\square$

## THE METHOD OF ISOCLINES

The basic idea in this method is that of isoclines. Consider the dynamics in (2.1). At a point  $(x_1, x_2)$  in the phase plane, the slope of the tangent to the trajectory can be determined by (2.5). An isocline is defined to be the locus of the points with a given tangent slope. An isocline with slope  $\alpha$  is thus defined to be

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = \alpha$$

This is to say that points on the curve

$$f_2(x_1, x_2) = \alpha f_1(x_1, x_2)$$

all have the same tangent slope  $\alpha$ .

In the method of isoclines, the phase portrait of a system is generated in two steps. In the first step, a field of directions of tangents to the trajectories is obtained. In the second step, phase plane trajectories are formed from the field of directions.

Let us explain the isocline method on the mass-spring system in (2.2). The slope of the trajectories is easily seen to be

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2}$$

Therefore, the isocline equation for a slope  $\alpha$  is

$$x_1 + \alpha x_2 = 0$$

*i.e.*, a straight line. Along the line, we can draw a lot of short line segments with slope  $\alpha$ . By taking  $\alpha$  to be different values, a set of isoclines can be drawn, and a field of directions of tangents to trajectories are generated, as shown in Figure 2.7. To obtain trajectories from the field of directions, we assume that the tangent slopes are locally constant. Therefore, a trajectory starting from any point in the plane can be found by connecting a sequence of line segments.

Let us use the method of isoclines to study the Van der Pol equation, a nonlinear equation.

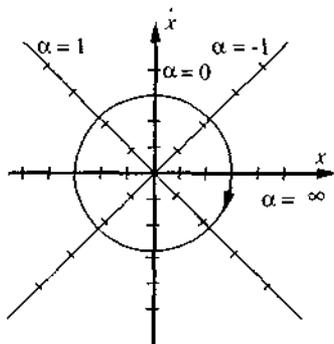


Figure 2.7 : Isoclines for the mass-spring system

### Example 2.6: The Van der Pol equation

For the Van der Pol equation

$$\ddot{x} + 0.2(x^2 - 1)\dot{x} + x = 0$$

an isocline of slope  $\alpha$  is defined by

$$\frac{d\dot{x}}{dx} = -\frac{0.2(x^2 - 1)\dot{x} + x}{\dot{x}} = \alpha$$

Therefore, the points on the curve

$$0.2(x^2 - 1)\dot{x} + x + \alpha\dot{x} = 0$$

all have the same slope  $\alpha$ .

By taking  $\alpha$  of different values, different isoclines can be obtained, as plotted in Figure 2.8. Short line segments are drawn on the isoclines to generate a field of tangent directions. The phase portraits can then be obtained, as shown in the plot. It is interesting to note that there exists a closed curve in the portrait, and the trajectories starting from both outside and inside converge to this curve. This closed curve corresponds to a limit cycle, as will be discussed further in section 2.5.  $\square$

Note that the same scales should be used for the  $x_1$  axis and  $x_2$  axis of the phase plane, so that the derivative  $dx_2/dx_1$  equals the geometric slope of the trajectories. Also note that, since in the second step of phase portrait construction we essentially assume that the slope of the phase plane trajectories is locally constant, more isoclines should be plotted in regions where the slope varies quickly, to improve accuracy.

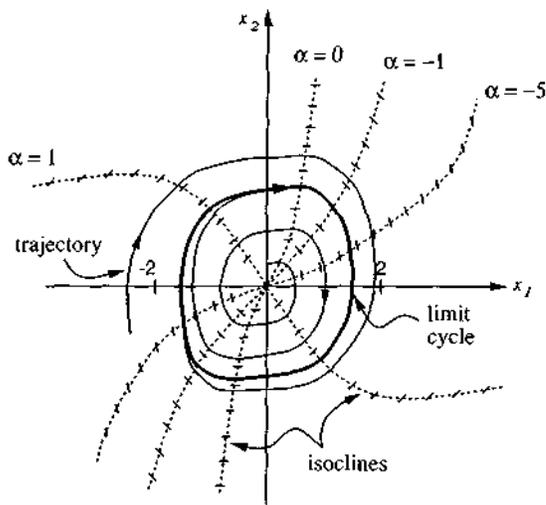


Figure 2.8 : Phase portrait of the Van der Pol equation

## 2.3 Determining Time from Phase Portraits

Note that time  $t$  does not explicitly appear in the phase plane having  $x_1$  and  $x_2$  as coordinates. However, in some cases, we might be interested in the time information. For example, one might want to know the time history of the system states starting from a specific initial point. Another relevant situation is when one wants to know how long it takes for the system to move from a point to another point in a phase plane trajectory. We now describe two techniques for computing time history from phase portraits. Both techniques involve a step-by-step procedure for recovering time.

### Obtaining time from $\Delta t \approx \Delta x / \dot{x}$

In a short time  $\Delta t$ , the change of  $x$  is approximately

$$\Delta x \approx \dot{x} \Delta t \quad (2.8)$$

where  $\dot{x}$  is the velocity corresponding to the increment  $\Delta x$ . Note that for a  $\Delta x$  of finite magnitude, the average value of velocity during a time increment should be used to improve accuracy. From (2.8), the length of time corresponding to the increment  $\Delta x$

is

$$\Delta t \approx \frac{\Delta x}{\dot{x}}$$

The above reasoning implies that, in order to obtain the time corresponding to the motion from one point to another point along a trajectory, one should divide the corresponding part of the trajectory into a number of small segments (not necessarily equally spaced), find the time associated with each segment, and then add up the results. To obtain the time history of states corresponding to a certain initial condition, one simply computes the time  $t$  for each point on the phase trajectory, and then plots  $x$  with respect to  $t$  and  $\dot{x}$  with respect to  $t$ .

### Obtaining time from $t = \int (1/\dot{x}) dx$

Since  $\dot{x} = dx/dt$ , we can write  $dt = dx/\dot{x}$ . Therefore,

$$t - t_0 = \int_{x_0}^x (1/\dot{x}) dx$$

where  $x$  corresponds to time  $t$  and  $x_0$  corresponds to time  $t_0$ . This equation implies that, if we plot a phase plane portrait with new coordinates  $x$  and  $(1/\dot{x})$ , then the area under the resulting curve is the corresponding time interval.

## 2.4 Phase Plane Analysis of Linear Systems

In this section, we describe the phase plane analysis of linear systems. Besides allowing us to visually observe the motion patterns of linear systems, this will also help the development of nonlinear system analysis in the next section, because a nonlinear system behaves similarly to a linear system around each equilibrium point.

The general form of a linear second-order system is

$$\dot{x}_1 = ax_1 + bx_2 \quad (2.9a)$$

$$\dot{x}_2 = cx_1 + dx_2 \quad (2.9b)$$

To facilitate later discussions, let us transform this equation into a scalar second-order differential equation. Note from (2.9a) and (2.9b) that

$$b\dot{x}_2 = b cx_1 + d(\dot{x}_1 - ax_1)$$

Consequently, differentiation of (2.9a) and then substitution of (2.9b) leads to

$$\ddot{x}_1 = (a+d)\dot{x}_1 + (cb-ad)x_1$$

Therefore, we will simply consider the second-order linear system described by

$$\ddot{x} + a\dot{x} + bx = 0 \quad (2.10)$$

To obtain the phase portrait of this linear system, we first solve for the time history

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \quad \text{for } \lambda_1 \neq \lambda_2 \quad (2.11a)$$

$$x(t) = k_1 e^{\lambda_1 t} + k_2 t e^{\lambda_1 t} \quad \text{for } \lambda_1 = \lambda_2 \quad (2.11b)$$

where the constants  $\lambda_1$  and  $\lambda_2$  are the solutions of the characteristic equation

$$s^2 + as + b = (s - \lambda_1)(s - \lambda_2) = 0$$

The roots  $\lambda_1$  and  $\lambda_2$  can be explicitly represented as

$$\lambda_1 = (-a + \sqrt{a^2 - 4b})/2 \quad \lambda_2 = (-a - \sqrt{a^2 - 4b})/2$$

For linear systems described by (2.10), there is only one singular point (assuming  $b \neq 0$ ), namely the origin. However, the trajectories in the vicinity of this singularity point can display quite different characteristics, depending on the values of  $a$  and  $b$ . The following cases can occur

1.  $\lambda_1$  and  $\lambda_2$  are both real and have the same sign (positive or negative)
2.  $\lambda_1$  and  $\lambda_2$  are both real and have opposite signs
3.  $\lambda_1$  and  $\lambda_2$  are complex conjugate with non-zero real parts
4.  $\lambda_1$  and  $\lambda_2$  are complex conjugates with real parts equal to zero

We now briefly discuss each of the above four cases.

#### STABLE OR UNSTABLE NODE

The first case corresponds to a *node*. A node can be stable or unstable. If the eigenvalues are negative, the singularity point is called a *stable node* because both  $x(t)$  and  $\dot{x}(t)$  converge to zero exponentially, as shown in Figure 2.9(a). If both eigenvalues are positive, the point is called an *unstable node*, because both  $x(t)$  and  $\dot{x}(t)$  diverge from zero exponentially, as shown in Figure 2.9(b). Since the eigenvalues are real, there is no oscillation in the trajectories.

### SADDLE POINT

The second case (say  $\lambda_1 < 0$  and  $\lambda_2 > 0$ ) corresponds to a *saddle point* (Figure 2.9(c)). The phase portrait of the system has the interesting "saddle" shape shown in Figure 2.9(c). Because of the unstable pole  $\lambda_2$ , almost all of the system trajectories diverge to infinity. In this figure, one also observes two straight lines passing through the origin. The diverging line (with arrows pointing to infinity) corresponds to initial conditions which make  $k_2$  (i.e., the unstable component) equal zero. The converging straight line corresponds to initial conditions which make  $k_1$  equal zero.

### STABLE OR UNSTABLE FOCUS

The third case corresponds to a focus. A *stable focus* occurs when the real part of the eigenvalues is negative, which implies that  $x(t)$  and  $\dot{x}(t)$  both converge to zero. The system trajectories in the vicinity of a stable focus are depicted in Figure 2.9(d). Note that the trajectories encircle the origin one or more times before converging to it, unlike the situation for a stable node. If the real part of the eigenvalues is positive, then  $x(t)$  and  $\dot{x}(t)$  both diverge to infinity, and the singularity point is called an *unstable focus*. The trajectories corresponding to an unstable focus are sketched in Figure 2.9(e).

### CENTER POINT

The last case corresponds to a center point, as shown in Figure 2.9(f). The name comes from the fact that all trajectories are ellipses and the singularity point is the center of these ellipses. The phase portrait of the undamped mass-spring system belongs to this category.

Note that the stability characteristics of linear systems are uniquely determined by the nature of their singularity points. This, however, is not true for nonlinear systems.

## 2.5 Phase Plane Analysis of Nonlinear Systems

In discussing the phase plane analysis of nonlinear systems, two points should be kept in mind. Phase plane analysis of nonlinear systems is related to that of linear systems, because the local behavior of a nonlinear system can be approximated by the behavior of a linear system. Yet, nonlinear systems can display much more complicated patterns in the phase plane, such as multiple equilibrium points and limit cycles. We now discuss these points in more detail.

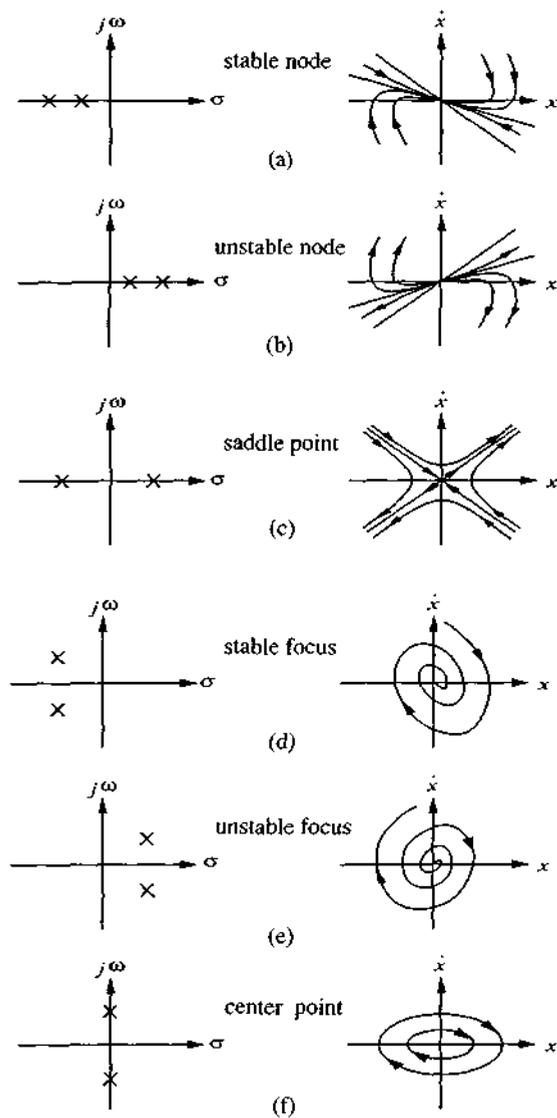


Figure 2.9 : Phase-portraits of linear systems

## LOCAL BEHAVIOR OF NONLINEAR SYSTEMS

In the phase portrait of Figure 2.2, one notes that, in contrast to linear systems, there are two singular points,  $(0, 0)$  and  $(-3, 0)$ . However, we also note that the features of the phase trajectories in the neighborhood of the two singular points look very much like those of linear systems, with the first point corresponding to a stable focus and the second to a saddle point. This similarity to a linear system in the local region of each singular point can be formalized by linearizing the nonlinear system, as we now discuss.

If the singular point of interest is not at the origin, by defining the difference between the original state and the singular point as a new set of state variables, one can always shift the singular point to the origin. Therefore, without loss of generality, we may simply consider Equation (2.1) with a singular point at 0. Using Taylor expansion, Equations (2.1a) and (2.1b) can be rewritten as

$$\dot{x}_1 = ax_1 + bx_2 + g_1(x_1, x_2)$$

$$\dot{x}_2 = cx_1 + dx_2 + g_2(x_1, x_2)$$

where  $g_1$  and  $g_2$  contain higher order terms.

In the vicinity of the origin, the higher order terms can be neglected, and therefore, the nonlinear system trajectories essentially satisfy the linearized equation

$$\dot{x}_1 = ax_1 + bx_2$$

$$\dot{x}_2 = cx_1 + dx_2$$

As a result, the local behavior of the nonlinear system can be approximated by the patterns shown in Figure 2.9.

## LIMIT CYCLES

In the phase portrait of the nonlinear Van der Pol equation, shown in Figure 2.8, one observes that the system has an unstable node at the origin. Furthermore, there is a closed curve in the phase portrait. Trajectories inside the curve and those outside the curve all tend to this curve, while a motion started on this curve will stay on it forever, circling periodically around the origin. This curve is an instance of the so-called "limit cycle" phenomenon. Limit cycles are unique features of nonlinear systems.

In the phase plane, a *limit cycle* is defined as an isolated closed curve. The trajectory has to be both closed, indicating the periodic nature of the motion, and isolated, indicating the limiting nature of the cycle (with nearby trajectories

converging or diverging from it). Thus, while there are many closed curves in the phase portraits of the mass-spring-damper system in Example 2.1 or the satellite system in Example 2.5, these are not considered limit cycles in this definition, because they are not isolated.

Depending on the motion patterns of the trajectories in the vicinity of the limit cycle, one can distinguish three kinds of limit cycles

1. **Stable Limit Cycles:** all trajectories in the vicinity of the limit cycle converge to it as  $t \rightarrow \infty$  (Figure 2.10(a));
2. **Unstable Limit Cycles:** all trajectories in the vicinity of the limit cycle diverge from it as  $t \rightarrow \infty$  (Figure 2.10(b));
3. **Semi-Stable Limit Cycles:** some of the trajectories in the vicinity converge to it, while the others diverge from it as  $t \rightarrow \infty$  (Figure 2.10(c));

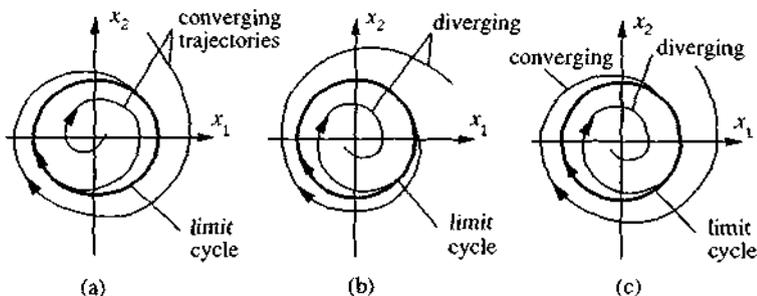


Figure 2.10 : Stable, unstable, and semi-stable limit cycles

As seen from the phase portrait of Figure 2.8, the limit cycle of the Van der Pol equation is clearly stable. Let us consider some additional examples of stable, unstable, and semi-stable limit cycles.

#### Example 2.7: stable, unstable, and semi-stable limit cycles

Consider the following nonlinear systems

$$(a) \quad \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1) \quad \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1) \quad (2.12)$$

$$(b) \quad \dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1) \quad \dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1) \quad (2.13)$$

$$(c) \quad \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \quad \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)^2 \quad (2.14)$$

Let us study system (a) first. By introducing polar coordinates

$$r = (x_1^2 + x_2^2)^{1/2} \quad \theta = \tan^{-1}(x_2/x_1)$$

the dynamic equations (2.12) are transformed as

$$\frac{dr}{dt} = -r(r^2 - 1) \quad \frac{d\theta}{dt} = -1$$

When the state starts on the unit circle, the above equation shows that  $\dot{r}(t) = 0$ . Therefore, the state will circle around the origin with a period  $1/2\pi$ . When  $r < 1$ , then  $\dot{r} > 0$ . This implies that the state tends to the circle from inside. When  $r > 1$ , then  $\dot{r} < 0$ . This implies that the state tends toward the unit circle from outside. Therefore, the unit circle is a stable limit cycle. This can also be concluded by examining the analytical solution of (2.12)

$$r(t) = \frac{1}{(1 + c_o e^{-2t})^{1/2}} \quad \theta(t) = \theta_o - t$$

where

$$c_o = \frac{1}{r_o^2} - 1$$

Similarly, one can find that the system (b) has an unstable limit cycle and system (c) has a semi-stable limit cycle.  $\square$

## 2.6 Existence of Limit Cycles

As mentioned in chapter 1, it is of great importance for control engineers to predict the existence of limit cycles in control systems. In this section, we state three simple classical theorems to that effect. These theorems are easy to understand and apply.

The first theorem to be presented reveals a simple relationship between the existence of a limit cycle and the number of singular points it encloses. In the statement of the theorem, we use  $N$  to represent the number of nodes, centers, and foci enclosed by a limit cycle, and  $S$  to represent the number of enclosed saddle points.

**Theorem 2.1 (Poincare)** *If a limit cycle exists in the second-order autonomous system (2.1), then  $N = S + 1$ .*

This theorem is sometimes called the *index theorem*. Its proof is mathematically involved (actually, a family of such proofs led to the development of algebraic topology) and shall be omitted here. One simple inference from this theorem is that a limit cycle must enclose at least one equilibrium point. The theorem's result can be

verified easily on Figures 2.8 and 2.10.

The second theorem is concerned with the asymptotic properties of the trajectories of second-order systems.

**Theorem 2.2 (Poincare-Bendixson)** *If a trajectory of the second-order autonomous system remains in a finite region  $\Omega$ , then one of the following is true:*

- (a) *the trajectory goes to an equilibrium point*
- (b) *the trajectory tends to an asymptotically stable limit cycle*
- (c) *the trajectory is itself a limit cycle*

While the proof of this theorem is also omitted here, its intuitive basis is easy to see, and can be verified on the previous phase portraits.

The third theorem provides a sufficient condition for the non-existence of limit cycles.

**Theorem 2.3 (Bendixson)** *For the nonlinear system (2.1), no limit cycle can exist in a region  $\Omega$  of the phase plane in which  $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$  does not vanish and does not change sign.*

**Proof:** Let us prove this theorem by contradiction. First note that, from (2.5), the equation

$$f_2 dx_1 - f_1 dx_2 = 0 \quad (2.15)$$

is satisfied for any system trajectories, including a limit cycle. Thus, along the closed curve  $L$  of a limit cycle, we have

$$\int_L (f_1 dx_2 - f_2 dx_1) = 0 \quad (2.16)$$

Using Stokes' Theorem in calculus, we have

$$\int_L (f_1 dx_2 - f_2 dx_1) = \iint \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 = 0$$

where the integration on the right-hand side is carried out on the area enclosed by the limit cycle.

By Equation (2.16), the left-hand side must equal zero. This, however, contradicts the fact that the right-hand side cannot equal zero because by hypothesis  $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$  does not vanish and does not change sign.  $\square$

Let us illustrate the result on an example.

**Example 2.8:** Consider the nonlinear system

$$\dot{x}_1 = g(x_2) + 4x_1x_2^2$$

$$\dot{x}_2 = h(x_1) + 4x_1^2x_2$$

Since

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 4(x_1^2 + x_2^2)$$

which is always strictly positive (except at the origin), the system does not have any limit cycles anywhere in the phase plane.  $\square$

The above three theorems represent very powerful results. It is important to notice, however, that they have no equivalent in higher-order systems, where exotic asymptotic behaviors other than equilibrium points and limit cycles can occur.

## 2.7 Summary

Phase plane analysis is a graphical method used to study second-order dynamic systems. The major advantage of the method is that it allows visual examination of the global behavior of systems. The major disadvantage is that it is mainly limited to second-order systems (although extensions to third-order systems are often achieved with the aid of computer graphics). The phenomena of multiple equilibrium points and of limit cycles are clearly seen in phase plane analysis. A number of useful classical theorems for the prediction of limit cycles in second-order systems are also presented.

## 2.8 Notes and References

Phase plane analysis is a very classical topic which has been addressed by numerous control texts. An extensive treatment can be found in [Graham and McRuer, 1961]. Examples 2.2 and 2.3 are adapted from [Ogata, 1970]. Examples 2.5 and 2.6 and section 2.6 are based on [Hsu and Meyer, 1968].

## 2.9 Exercises

2.1 Draw the phase portrait and discuss the properties of the linear, unity feedback control system of open-loop transfer function

$$G(p) = \frac{10}{p(1 + 0.1p)}$$

2.2 Draw the phase portraits of the following systems, using isoclines

$$(a) \quad \ddot{\theta} + \dot{\theta} + 0.5\theta = 0$$

$$(b) \quad \ddot{\theta} + \dot{\theta} + 0.5\theta = 1$$

$$(c) \quad \ddot{\theta} + \dot{\theta}^2 + 0.5\theta = 0$$

2.3 Consider the nonlinear system

$$\dot{x} = y + x(x^2 + y^2 - 1) \sin \frac{1}{x^2 + y^2 - 1}$$

$$\dot{y} = -x + y(x^2 + y^2 - 1) \sin \frac{1}{x^2 + y^2 - 1}$$

Without solving the above equations explicitly, show that the system has infinite number of limit cycles. Determine the stability of these limit cycles. (*Hint:* Use polar coordinates.)

2.4 The system shown in Figure 2.10 represents a satellite control system with rate feedback provided by a gyroscope. Draw the phase portrait of the system, and determine the system's stability.

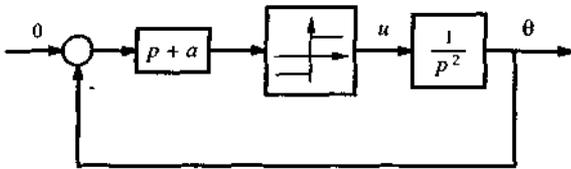


Figure 2.10 : Satellite control system with rate feedback