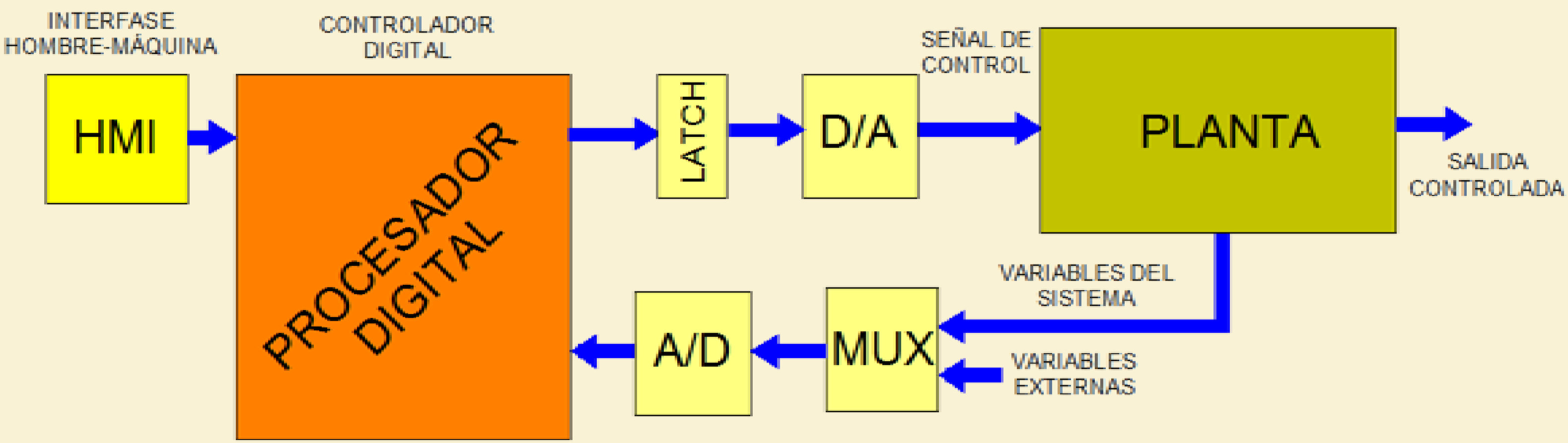
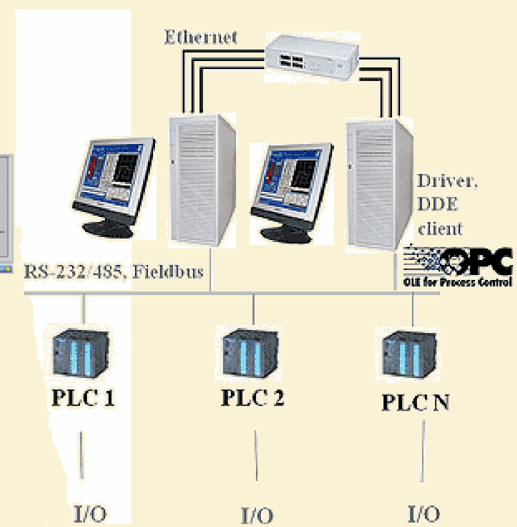
TEORÍA DE CONTROL

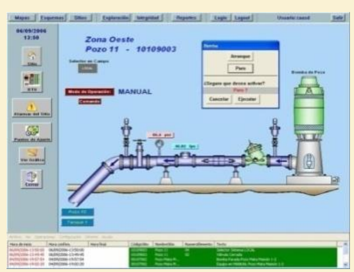
SISTEMAS DISCRETOS

SISTEMAS DISCRETOS

ARQUITECTURA DE UN SISTEMA DE CONTROL DIGITAL



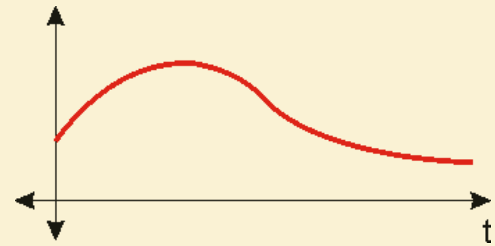




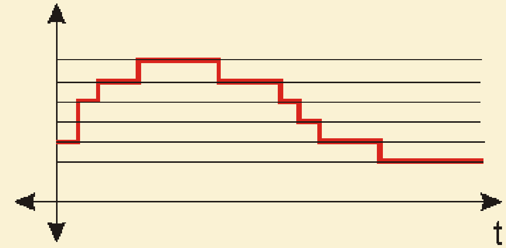
SISTEMAS DISCRETOS

TIPOS DE SEÑALES

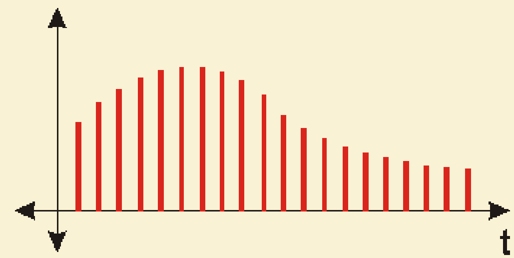
TIEMPO CONTINUO
- ANALÓGICAS



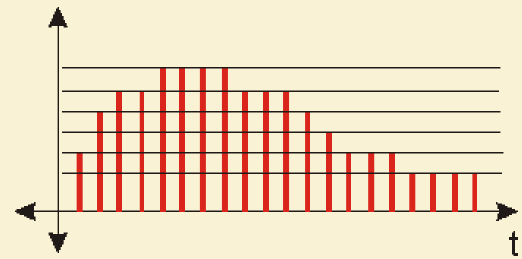
- CUANTIFICADAS



TIEMPO DISCRETO
- NO CUANTIFICADAS



- DIGITALES



TRANSFORMADA Z

$$X(z) = Z\{x(t)\} = Z\{x(kT)\} = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

$$X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + x(3T)z^{-3} + ... + x(kT)z^{-k} + ...$$

ESCALÓN UNITARIO

$$x(t) = \begin{cases} 1 \quad \text{para } t \geq 0 \\ 0 \quad \text{para } t < 0 \end{cases}$$

$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + ... \rightarrow \frac{1}{1-r} = \frac{1}{1-z^{-1}}$$

$$a^k$$

$$x(k) = \begin{cases} a^k \quad \text{para } k \geq 0 \\ 0 \quad \text{para } k < 0 \end{cases}$$

$$X(z) = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + ... \rightarrow r = az^{-1}$$

$$X(z) = \frac{1}{(1-az^{-1})}$$

TRANSFORMADA Z

e^{-at}

$$x(t)=\begin{cases} e^{-at} & \text{para } t \geq 0 \\ 0 & \text{para } t < 0 \end{cases}$$

$$X(z)=1+e^{-aT}z^{-1}+e^{-2aT}z^{-2}+e^{-3aT}z^{-3}+\ldots \rightarrow r=e^{-aT}z^{-1}$$

$$X(z)=\frac{1}{\left(1-e^{-aT}z^{-1}\right)}$$

RAMPA UNITARIA

$$x(t)=\begin{cases} t & \text{para } t \geq 0 \\ 0 & \text{para } t < 0 \end{cases}$$

$$X(z)=Tz^{-1}+2Tz^{-2}+3Tz^{-3}+\ldots=$$

$$=Tz^{-1}\left(1+2z^{-1}+3z^{-2}+4z^{-3}+\ldots\right)=\frac{Tz^{-1}}{\left(1-z^{-1}\right)^2}$$

TRANSFORMADA Z

PROPIEDADES

$$Z\{a\ x(t)\} = a\ X(z)$$

$$Z\{a\,x(t) + b\,y(t)\} = a\,X(z) + b\,Y(z)$$

$$Z\{a^k\ x(t)\} = X(a^{-1}z)$$

$$Z\{x(t+nT)\} = z^n\left(X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right)$$

$$Z\{x(t-nT)\} = z^{-n}X(z)$$

Teorema del Valor inicial

$$f(0) = \lim_{|z|\to\infty} F(z)$$

Teorema del Valor Final

$$f(\infty) = \lim_{z\to 1} (z-1)F(z)$$

SISTEMAS DISCRETOS

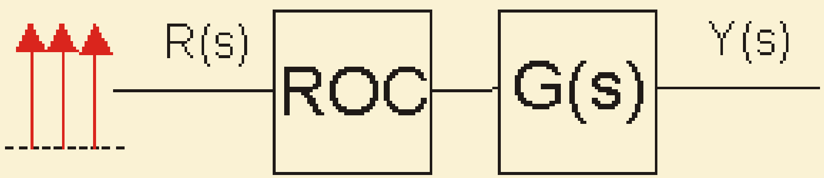
RETENCIÓN DE ORDEN CERO

$$y(t) = u(t) - u(t-T)$$

$$u(t) = \text{escalón}$$

$$y(t) = \text{respuesta al impulso}$$

$$\frac{Y(s)}{R(s)} = \left[\frac{1}{s} - \frac{1}{s}e^{-sT}\right] = \frac{1-e^{-sT}}{s} = \text{ROC}$$



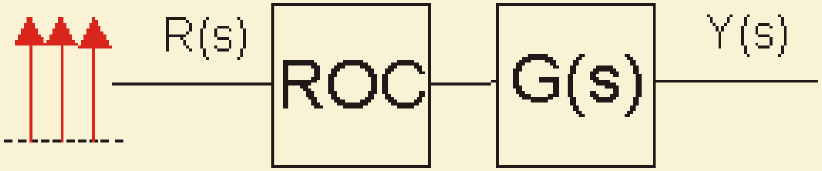
$$\frac{Y(s)}{R(s)} = \frac{1-e^{-sT}}{s}G(s) = \left(1-e^{-sT}\right)\, G_1(s)$$

$$X_1(s) = e^{-sT}G_1(s)$$

$$x_1(t) = \int_0^t g_0(t-\tau)\, g_1(\tau)\, d\tau \qquad \mathrm{g}_0(t) = \delta(t-T)$$

$$x_1(t) = \int_0^t \delta(t-T-\tau)\, g_1(\tau)\, d\tau = \mathrm{g}_1(t-T)$$

RETENCIÓN DE ORDEN CERO



$$Z\{g_1(t)\} = G_1(z)$$

$$\frac{Y(s)}{R(s)} = \frac{1-e^{-sT}}{s} G(s)$$

$$Z\{x_1(t)\} = Z\{g_1(t-T)\} = z^{-1}G_1(z)$$

$$\frac{Y(z)}{R(z)} = Z\{G_1(s) - e^{-sT}G_1(s)\}$$

$$\frac{Y(z)}{R(z)} = G_1(z) - z^{-1}G_1(z)$$

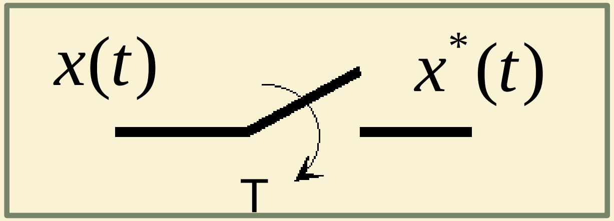
$$\frac{Y(z)}{R(z)} = (1-z^{-1})G_1(z)$$

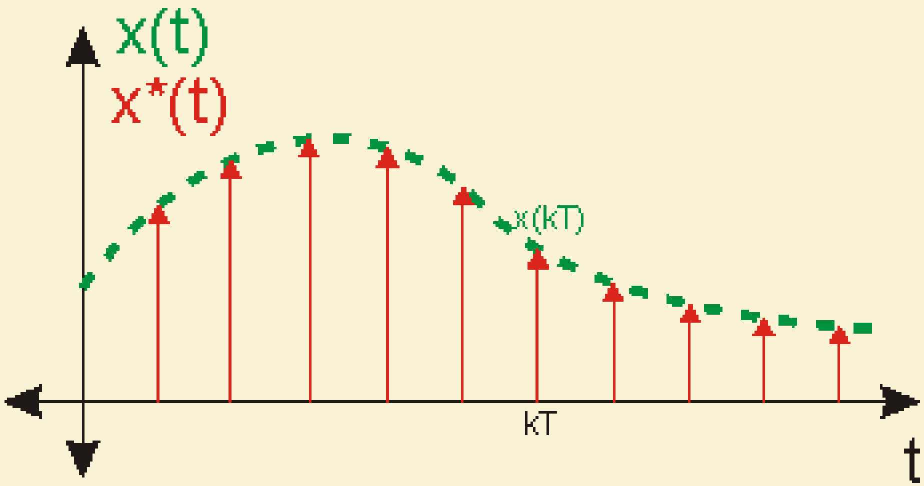
$$\frac{Y(z)}{R(z)} = (1-z^{-1})\ Z\left\{\frac{G(s)}{s}\right\}$$

MUESTREO MEDIANTE IMPULSOS

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT)$$

$$\delta_T(t-kT) = \sum_{k=0}^{\infty} \delta(t-kT)$$





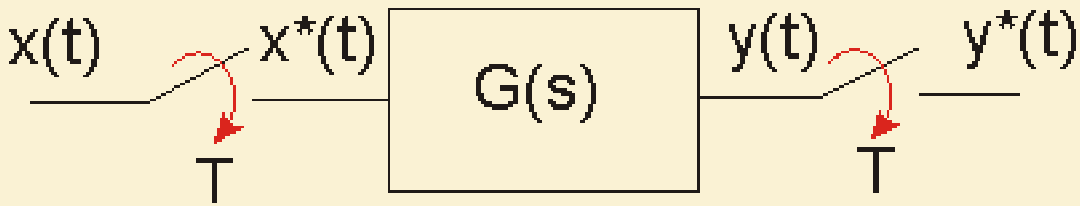
$$X^*(s) = x(0)\mathcal{L}\{\delta(t)\} + x(T)\mathcal{L}\{\delta(t-T)\} + \ldots$$

$$X^*(s) = x(0) + x(T)e^{-sT} + x(2T)e^{-2sT} + \ldots$$

$$X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-skT} = X(z) \quad \text{con} \quad z = e^{sT}$$

SISTEMAS DISCRETOS

FUNCIÓN TRANSFERENCIA DE PULSO



$$Z\{y(t)\} = Y(z) = \sum_{k=0}^{\infty} y(kT) z^{-k}$$

$$y(t) = \int_0^t g(t-\tau)\, x^*(\tau)\, d\tau = \int_0^t x^*(t-\tau)\, g(\tau)\, d\tau$$

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT)$$

La respuesta de y(t) es la suma de las respuestas al impulso

$$y(t) = \begin{cases} g(t)x(0) & 0 \le t < T \\ g(t)x(0) + g(t-T)x(T) & \mathrm{T} \le t < 2T \\ \ldots & \\ g(t)x(0) + g(t-T)x(T) + \ldots + g(t-kT)x(kT) + \ldots & kT \le t < (k+1)T \end{cases}$$

FUNCIÓN TRANSFERENCIA DE PULSO

$$y(t) = \sum_{h=0}^{\infty} g(t - hT)\, x(hT)$$

$$y(kT) = \sum_{h=0}^{\infty} g(kT - hT)\, x(hT)$$

Sumatoria de Convolución

$$y(kT) = x(kT)_{*} g(kT)$$

$$Y(z) = \sum_{k=0}^{\infty} y(kT) z^{-k}$$

$$Y(z) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} g(kT - hT) x(hT) z^{-k}$$

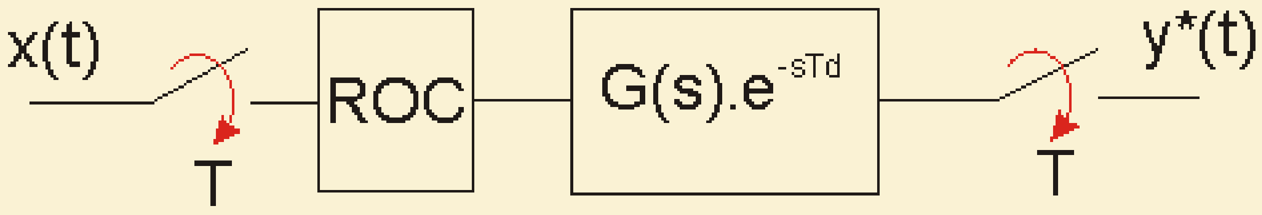
$$m = k - h$$

$$Y(z) = \sum_{m=0}^{\infty} \sum_{h=0}^{\infty} g(mT) x(hT) z^{-m-h}$$

$$Y(z) = \sum_{m=0}^{\infty} g(mT)\, z^{-m} . \sum_{h=0}^{\infty} x(hT)\, z^{-h}$$

$$Y(z) = G(z).\ X(z)$$

SISTEMAS CON RETARDO



$$\frac{Y(z)}{X(z)} = (1 - z^{-1}) \ Z\left\{\frac{G(s)}{s} e^{-sT_d}\right\}$$

$$\frac{Y(z)}{X(z)} = (1 - z^{-1}) \ z^{-\frac{T_d}{T}} \ Z\left\{\frac{G(s)}{s}\right\}$$

Si Td = N.T

$$\frac{Y(z)}{X(z)} = (1 - z^{-1}) \ z^{-N} \ Z\left\{\frac{G(s)}{s}\right\}$$

SISTEMAS DISCRETOS

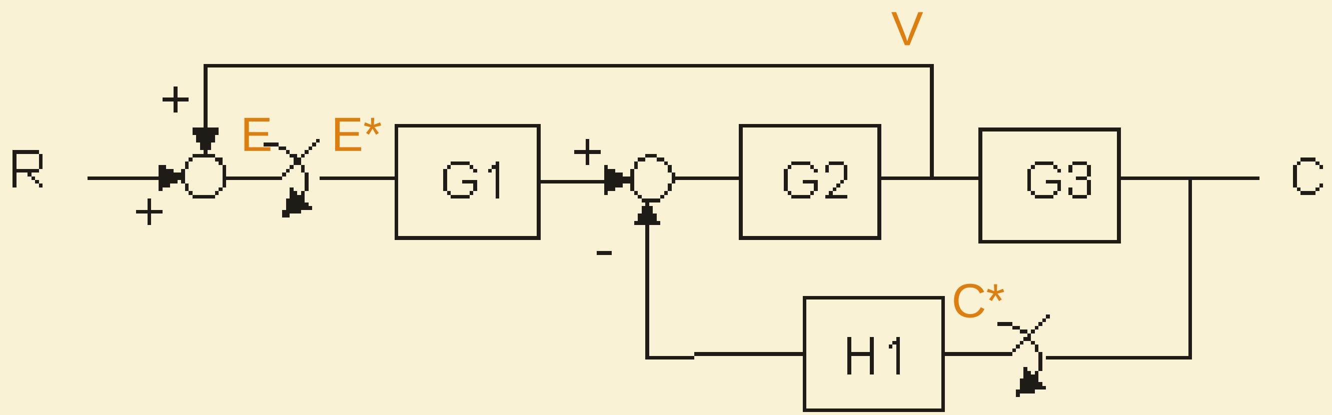
TABLA DE TRANSFORMADA Z

Si bien la transformada Z se define en el dominio temporal existen tablas que permiten transformar al plano Z expresiones representadas en el campo transformado de Laplace.

$f(t)$ F. Continua	$f(kT)$ F. Discreta, muestreada	$F(s)$ Transforma de Laplace	$F(Z)$ Transformada Z
$\delta(t)$ Impulso de Dirac	$\delta(kT)$	1	1
$\delta(t-T)$	$\delta(t-kT)^*$	e^{-TS}	
$\upsilon(t)$ Escalón Unitario	$\upsilon(kT)$	$\frac{1}{S}$	$\frac{Z}{Z-1}$
$\upsilon(t-T)$	$\upsilon(t-kT)^*$	$\frac{e^{-TS}}{S}$	
	$\upsilon(t-kT-T\omega)$	$\frac{e^{-SkT-T\omega S}}{S}$	
t Rampa	kT	$\frac{1}{S^2}$	$\frac{Tz}{(z-1)^2}$
t^2	$(kT)^2$	$\frac{2}{S^3}$	$\frac{T^2 z(z+1)}{(z-1)^3}$
t^3	$(kT)^3$	$\frac{6}{S^4}$	$\frac{T^3 z(z^2+4z+1)}{(z-1)^4}$
$\frac{1}{2}t^2$	$\frac{1}{2}(kT)^2$	$\frac{1}{S^3}$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
t^{m-1}; $m=1,2,3,\ldots$		$\frac{(m-1)!}{s^m}$	$\lim_{h\to 0}\left[(-1)^{m-1}\left(\frac{\partial^{m-1}\frac{z}{z-e^{-Th}}}{\partial b^{m-1}}\right)\right]$
e^{-at}	e^{-akT}	$\frac{1}{S+a}$	$\frac{Z}{Z-e^{-aT}}$
te^{-at}	kTe^{-akT}	$\frac{1}{(S+a)^2}$	$\frac{Te^{-aT}z}{(z-e^{-aT})^2}$
t^2e^{-at}	$(kT)^2 e^{-akT}$	$\frac{2}{(S+a)^3}$	$\frac{T^2e^{-aT}z(z+e^{-aT})}{(z-e^{-aT})^3}$
$\sin(bt)$	$\sin(bkT)$	$\frac{b}{s^2+b^2}$	$\frac{z\sin(bT)}{z^2-2z\cos(bT)+1}$
$\cos(bt)$	$\cos(bkT)$	$\frac{S}{S^2+b^2}$	$\frac{z^2-z\sin(bT)}{z^2-2z\cos(bT)+1}$
$e^{-at}\sin(bt)$	$e^{-akT}\sin(bkT)$	$\frac{b}{(S+a)^2+b^2}$	$\frac{ze^{-aT}\sin(bT)}{z^2-2ze^{-aT}\cos(bT)+e^{-2aT}}$
$e^{-at}\cos(bt)$	$e^{-akT}\cos(bkT)$	$\frac{S+a}{(S+a)^2+b^2}$	$\frac{z^2-ze^{-aT}\cos(bT)}{z^2-2ze^{-aT}\cos(bT)+e^{-2aT}}$

EJEMPLO:

El siguientes diagrama representa un sistema muestreado de control. Obtenga una expresión para la salida C(z) cuando se le aplica una entrada R(s) en forma de escalón de amplitud unitaria.



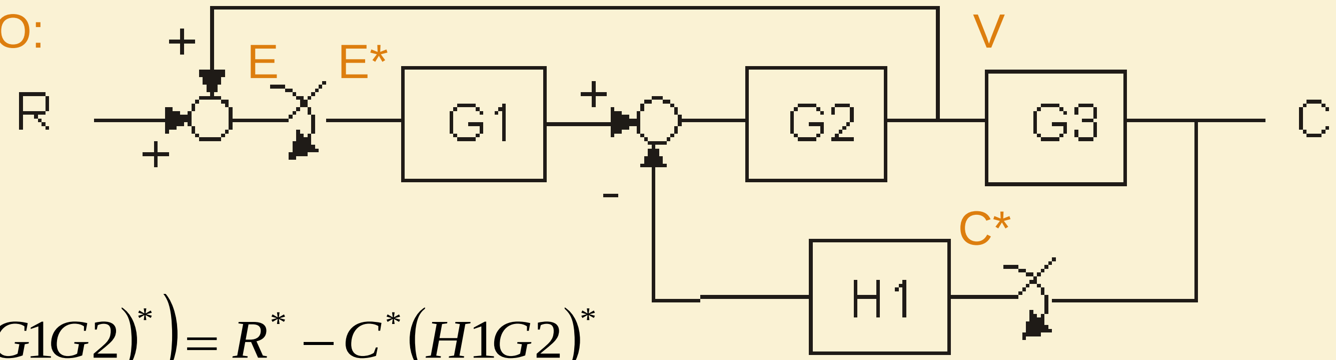
$$E = R + V = R + (E^* G1 - C^* H1)G2 = R + E^* G1G2 - C^* H1G2$$

$$E^* = R^* + E^* (G1G2)^* - C^* (H1G2)^*$$

$$E^* \left(1 - (G1G2)^*\right) = R^* - C^* (H1G2)^*$$

SISTEMAS DISCRETOS

EJEMPLO:



$$E^*\left(1-(G1G2)^*\right)=R^*-C^*(H1G2)^*$$

$$E^*=\frac{R^*-C^*(H1G2)^*}{\left(1-(G1G2)^*\right)}$$

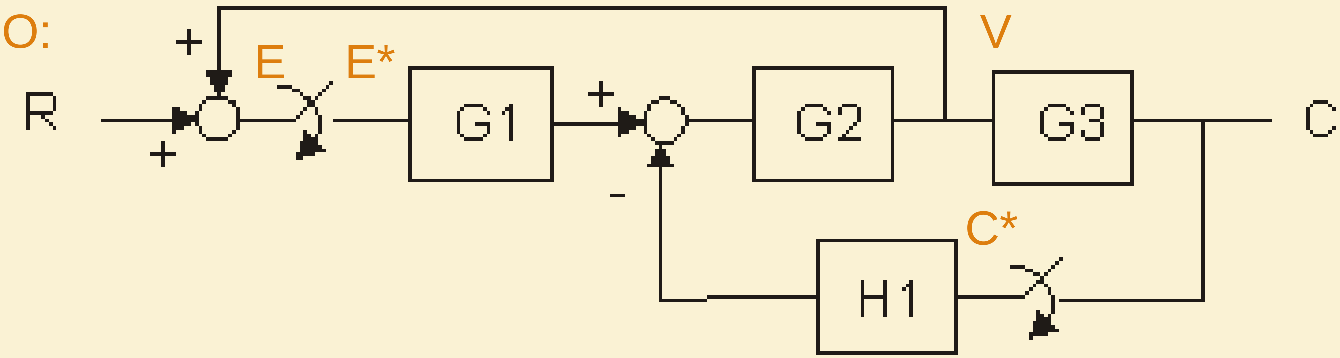
$$C=(E^*G1-C^*H1)G2G3=E^*G1G2G3-C^*H1G2G3$$

$$C^*=E^*(G1G2G3)^*-C^*(H1G2G3)^*$$

$$C^*=\frac{R^*-C^*(H1G2)^*}{\left(1-(G1G2)^*\right)}(G1G2G3)^*-C^*(H1G2G3)^*$$

SISTEMAS DISCRETOS

EJEMPLO:



$$C^* = \frac{R^*(G1G2G3)^* - C^*(H1G2)^*(G1G2G3)^* - C^*(H1G2G3)^*\left(1-(G1G2)^*\right)}{\left(1-(G1G2)^*\right)}$$

$$C^*\left(1-(G1G2)^*\right) = R^*(G1G2G3)^* - C^*(H1G2)^*(G1G2G3)^* - C^*(H1G2G3)^*\left(1-(G1G2)^*\right)$$

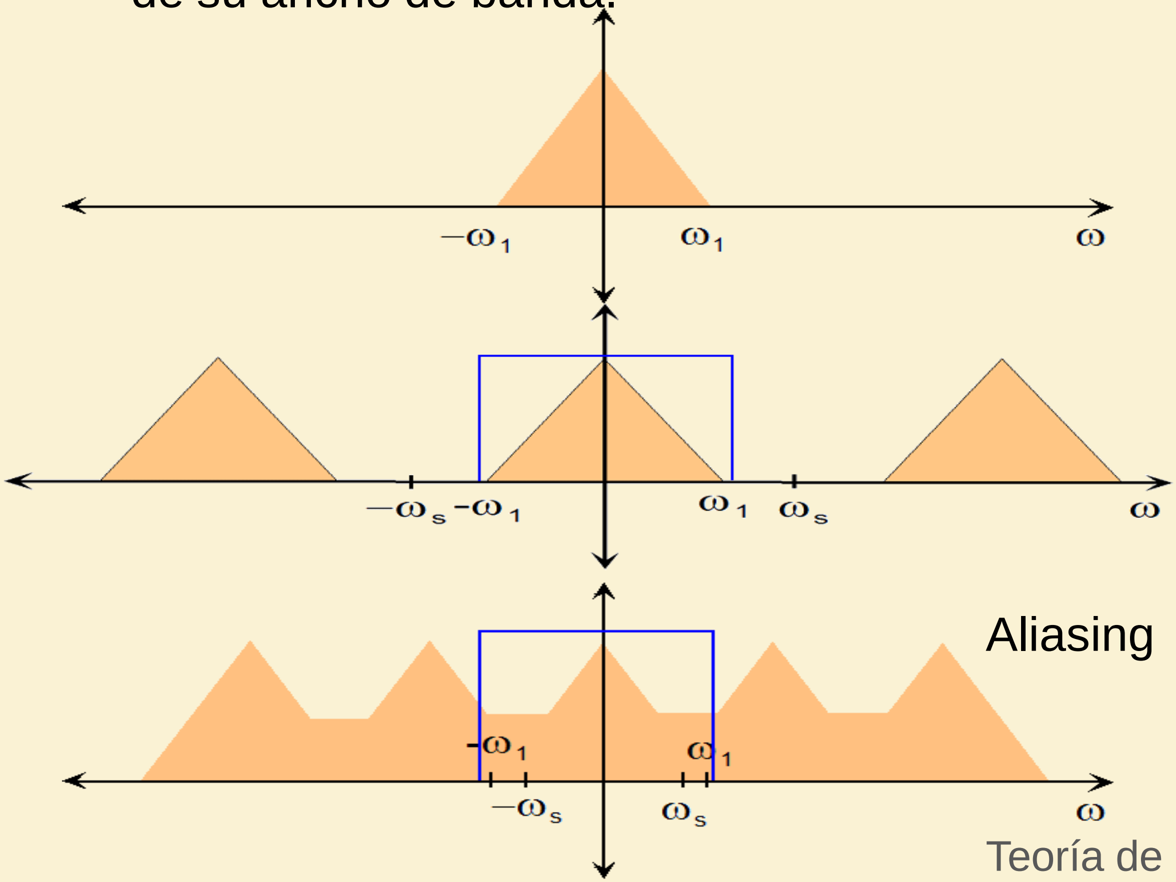
$$C^*\left(1-(G1G2)^* + (H1G2)^*(G1G2G3)^* + (H1G2G3)^*\left(1-(G1G2)^*\right)\right) = R^*(G1G2G3)^*$$

$$C^* = \frac{R^*(G1G2G3)^*}{\left(1-(G1G2)^* + (H1G2)^*(G1G2G3)^* + (H1G2G3)^*\left(1-(G1G2)^*\right)\right)}$$

$$C(z) = \frac{R(z)(G1G2G3)(z)}{\left(1-(G1G2)(z) + (H1G2)(z)(G1G2G3)(z) + (H1G2G3)(z)\left(1-(G1G2)(z)\right)\right)}$$

Teorema del muestreo

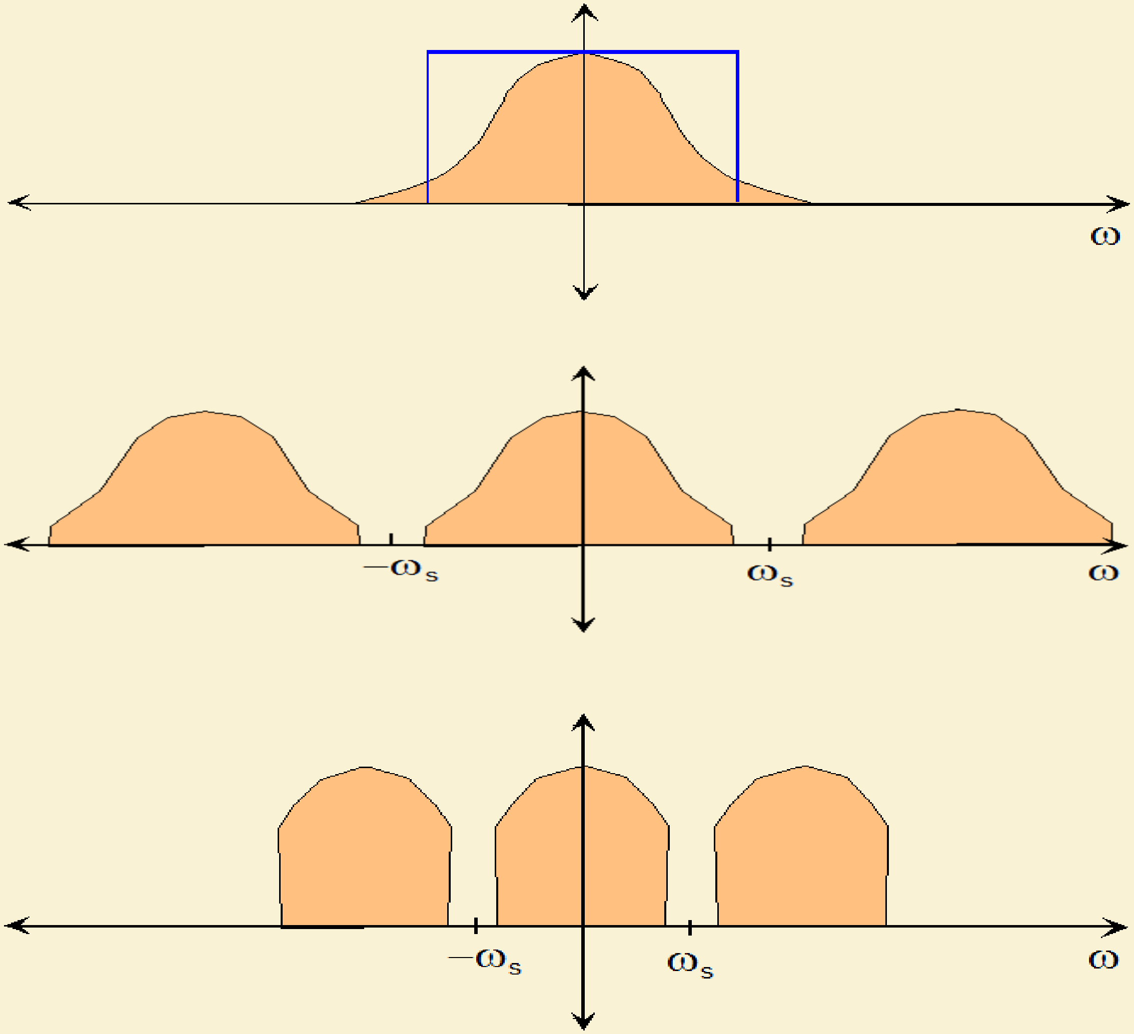
El teorema demuestra que la reconstrucción exacta de una señal periódica continua en banda base a partir de sus muestras, es matemáticamente posible si la señal está limitada en banda y la tasa de muestreo es superior al doble de su ancho de banda.



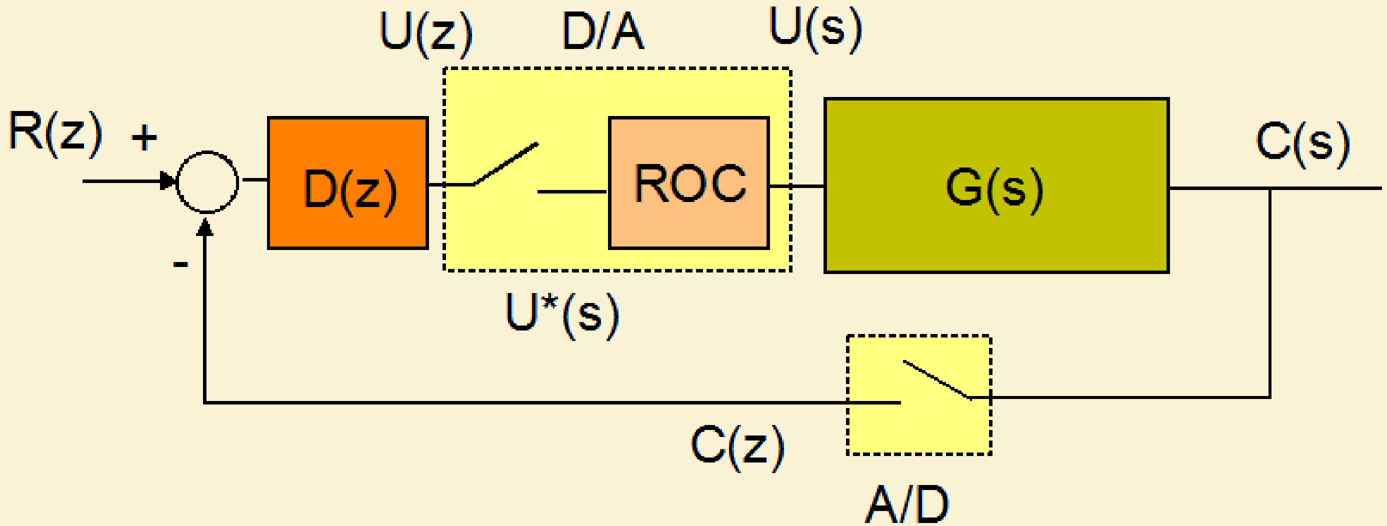
$$\omega_s > 2\omega_1$$

$$\omega_s > 2\omega_1$$

FILTRO ANTI ALIASING



CONTROL DISCRETO



$$\frac{C(s)}{U^*(s)} = \frac{1-e^{-sT}}{s}G(s)$$

$$G(z) = \frac{C(z)}{U(z)} = \left(1-z^{-1}\right)Z\left\{\frac{G(s)}{s}\right\}$$

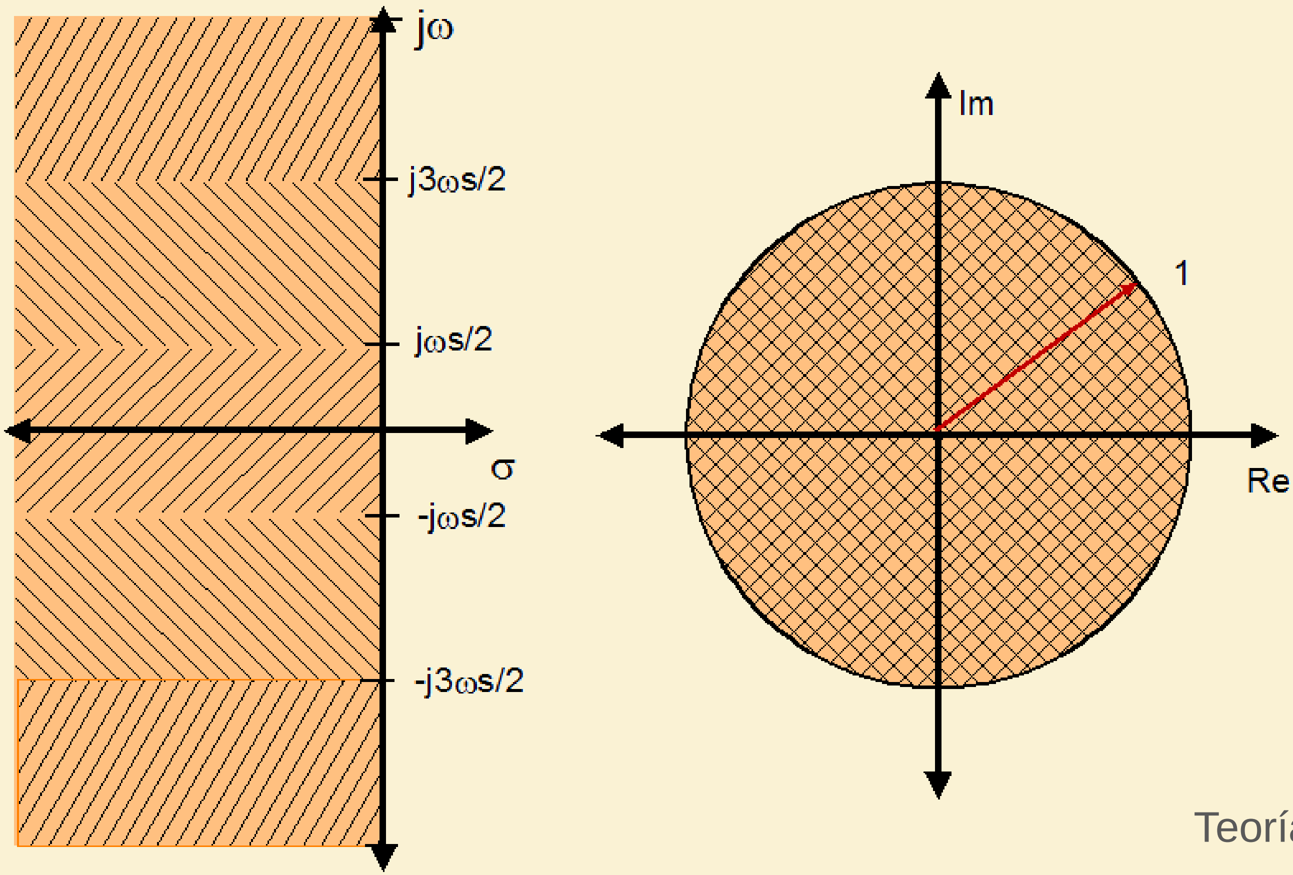
$$\frac{C(z)}{R(z)} = \frac{G(z)D(z)}{1+G(z)D(z)}$$

CORRESPONDENCIA ENTRE EL PLANO S Y EL PLANO Z

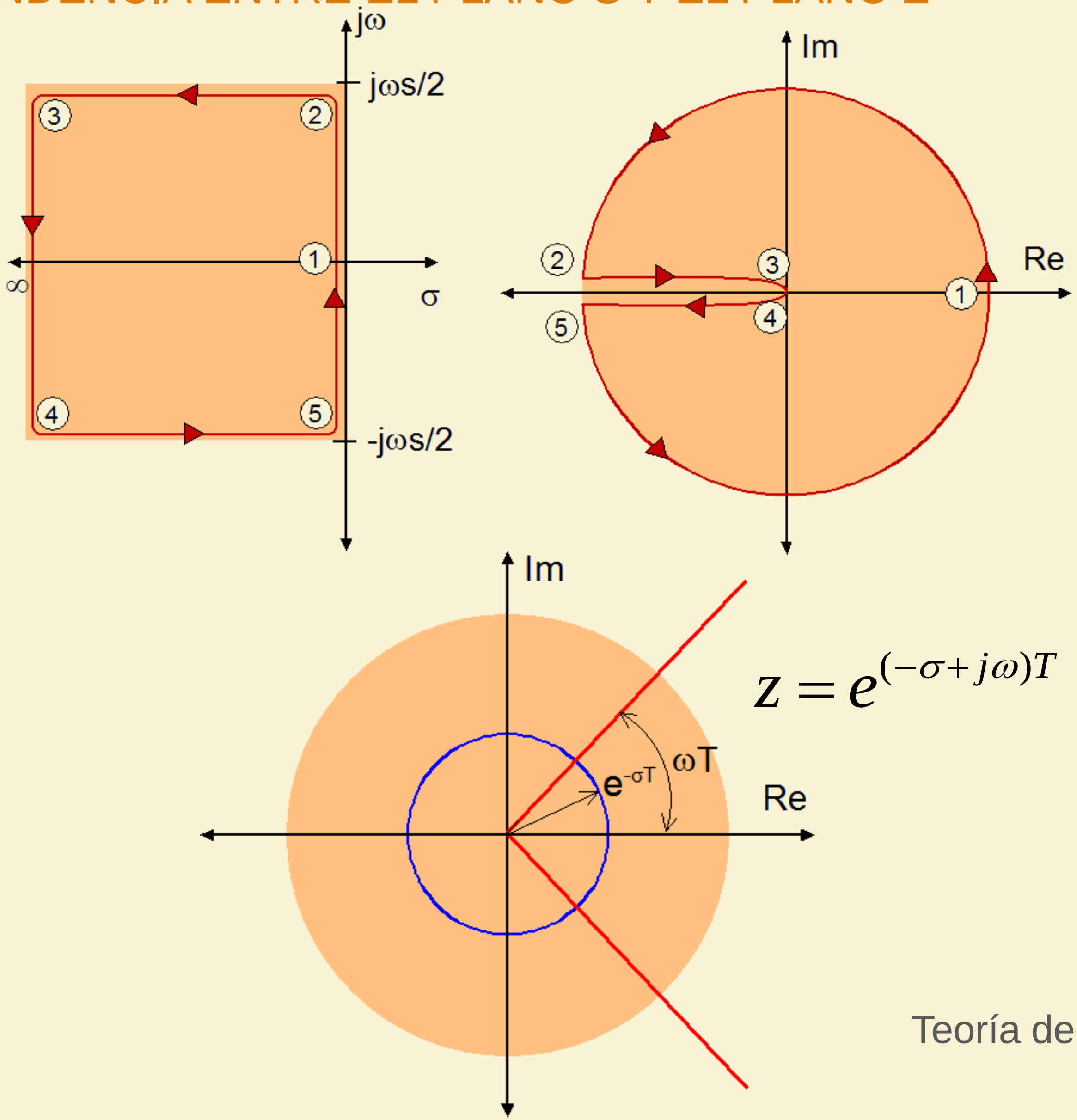
$z = e^{sT}$ con $s = \sigma + j\omega$

$$z = e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T} = e^{\sigma T} e^{j(\omega T + 2\pi k)}$$

$$|z| = e^{\sigma T} < 1$$

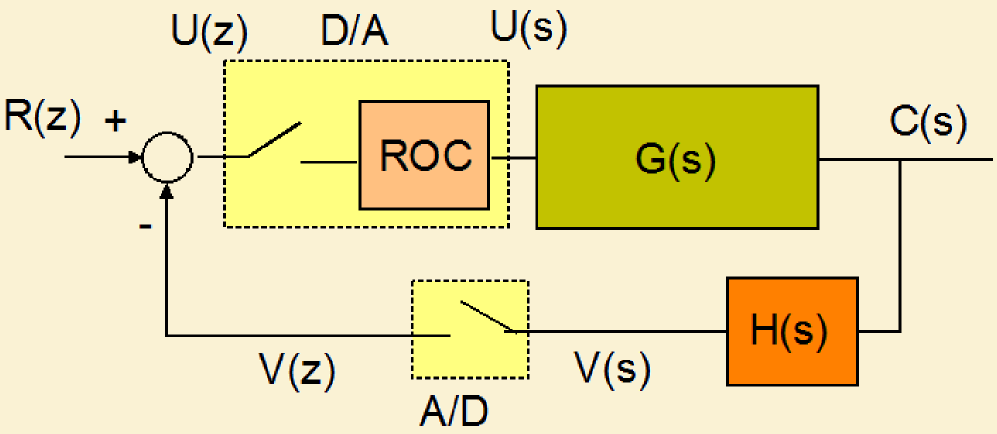


CORRESPONDENCIA ENTRE EL PLANO S Y EL PLANO Z



$$z = e^{(-\sigma + j\omega)T} = e^{-\sigma T} e^{j\omega T}$$

ANÁLISIS DE ESTABILIDAD



$$GH(z) = \frac{V(z)}{U(z)} = \left(1 - z^{-1}\right) Z\left\{\frac{GH(s)}{s}\right\}$$

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

$$P(z) = 1 + GH(z) = 0$$

$$G(s) = \frac{1}{s(s+1)} \quad H(s) = 1 \quad T = 1$$

$$G(z) = (1 - z^{-1}) Z\left\{\frac{1}{s^2} - \frac{1}{s} + \frac{1}{(s+1)}\right\}$$

$$G(z) = \frac{(z-1)}{z}\left[\frac{Tz}{(z-1)^2} - \frac{z}{(z-1)} + \frac{z}{(z - e^{-T})}\right]$$

ANÁLISIS DE ESTABILIDAD

$$G(z)=\frac{(z-1)}{z}\left[\frac{z\left(T\left(z-e^{-T}\right)-(z-1)\left(z-e^{-T}\right)+(z-1)^2\right)}{(z-1)^2\left(z-e^{-T}\right)}\right]$$

$$G(z)=\left[\frac{z\left(T+e^{-T}-1\right)+\left(1-e^{-T}-Te^{-T}\right)}{(z-1)\left(z-e^{-T}\right)}\right]$$

$$G(z)=\frac{0,3679\left(z+0,7183\right)}{\left(z-1\right)\left(z-0,3679\right)}$$

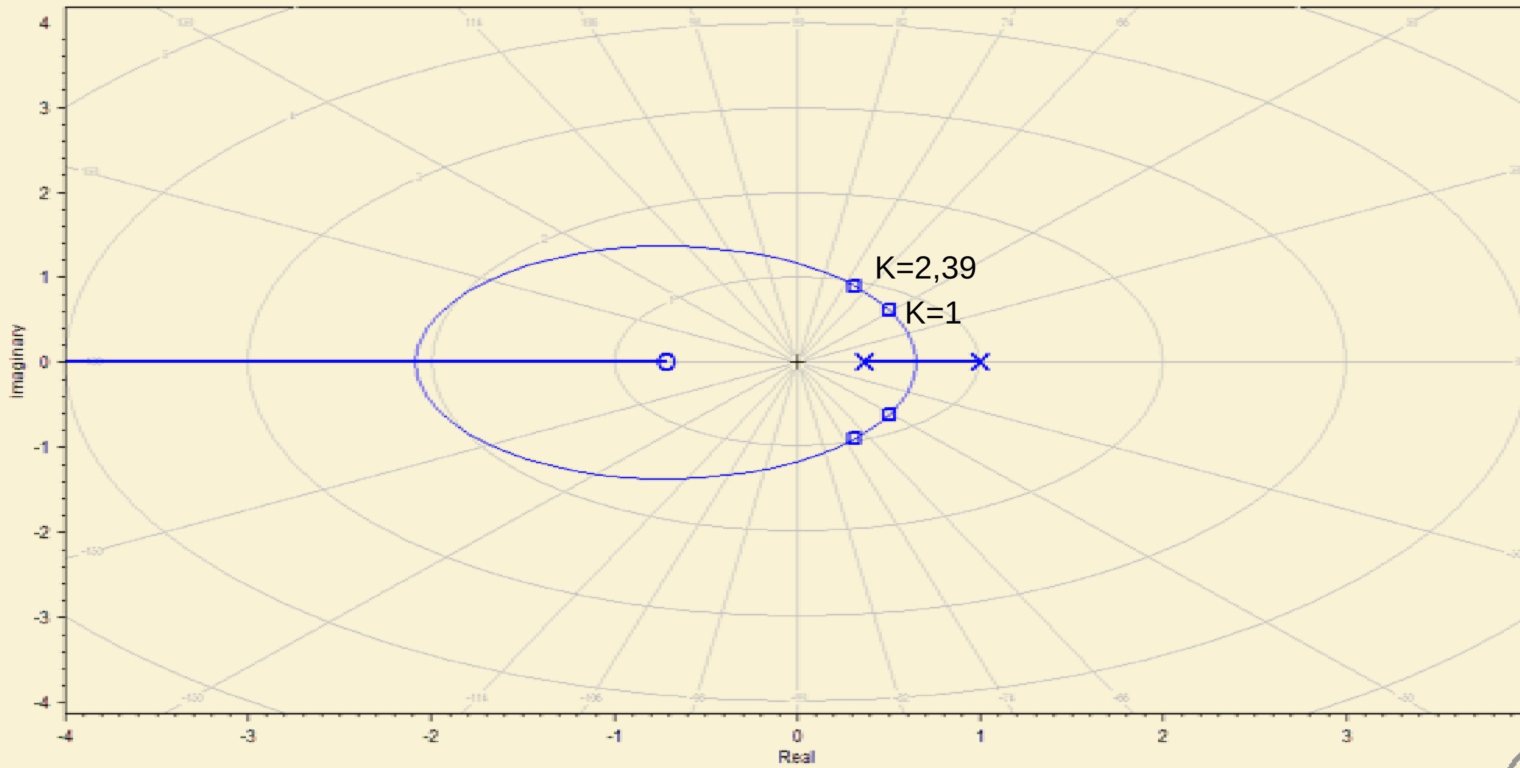
$$P(z)=\left(z-1\right)\left(z-0,3679\right)+0,3679z+0,2642=z^2-z+0,6321$$

$$z_1=0,5+j0,6182 \qquad z_2=0,5-j0,6182$$

$$|z_1|=|z_2|=0,7950919<1$$ ESTABLE

ANÁLISIS DE ESTABILIDAD

$$G(s)=\frac{K}{s(s+1)} \quad H(s)=1 \quad T=1 \qquad G(z)=\frac{0{,}3679K\left(z+0{,}7183\right)}{\left(z-1\right)\left(z-0{,}3679\right)}$$



ANÁLISIS DE ESTABILIDAD

Si el periodo de muestreo T no es conocido

$$G(z)=\left[\frac{z\left(T+e^{-T}-1\right)+\left(1-e^{-T}-Te^{-T}\right)}{(z-1)\left(z-e^{-T}\right)}\right]$$

$$1+G(z)=\frac{(z-1)\left(z-e^{-T}\right)+\left[z\left(T+e^{-T}-1\right)+\left(1-e^{-T}-Te^{-T}\right)\right]}{(z-1)\left(z-e^{-T}\right)}$$

$$1+G(z)=\frac{z^{2}+\left(T-2\right)z-Te^{-T}+1}{(z-1)\left(z-e^{-T}\right)}$$

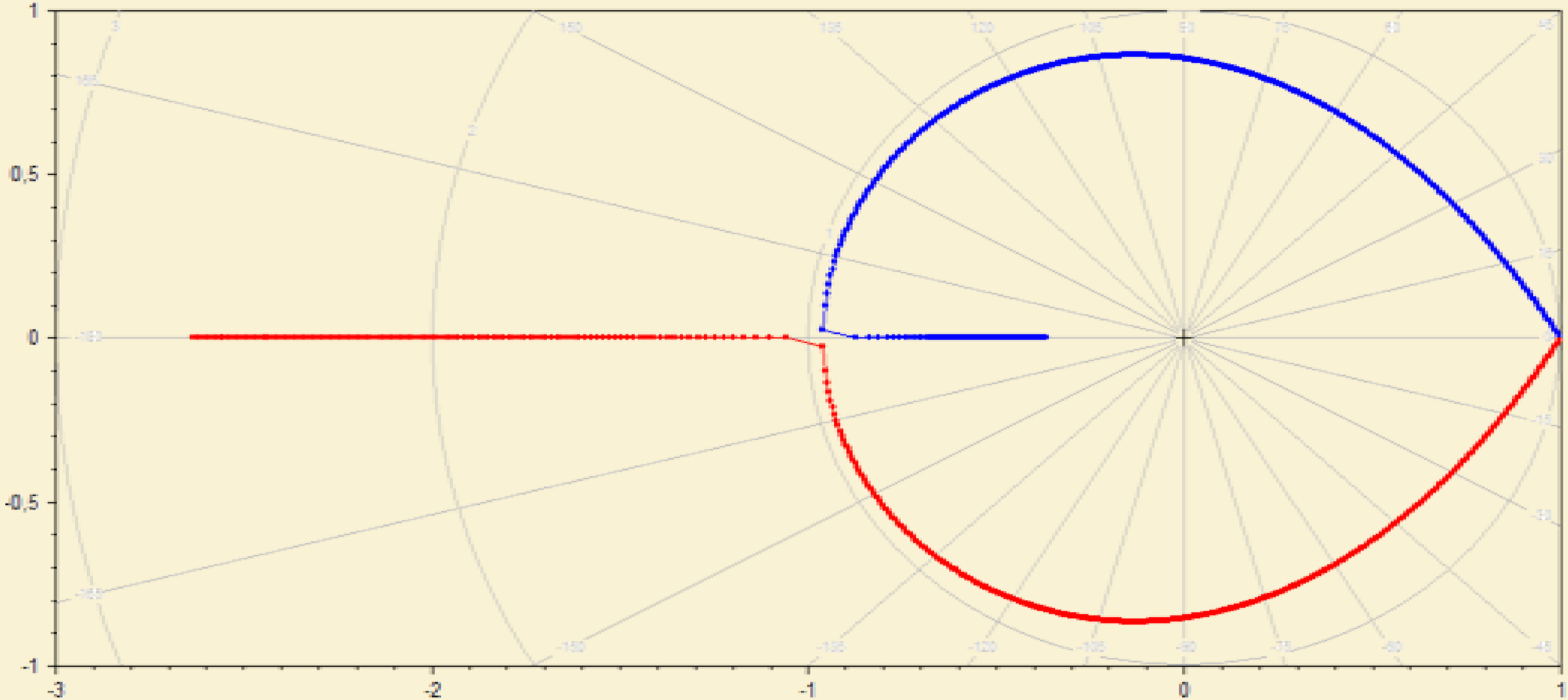
Para T=3.923

$$1+G(z)=\frac{z^{2}+1,923z+0,9224}{(z-1)\left(z-0,01978\right)}$$

Y los ceros son z1= -0,91583 y z2= -1,00716

ANÁLISIS DE ESTABILIDAD

$$1+G(z)=\frac{z^2+\left(T-2\right)z-Te^{-T}+1}{(z-1)\left(z-e^{-T}\right)}$$



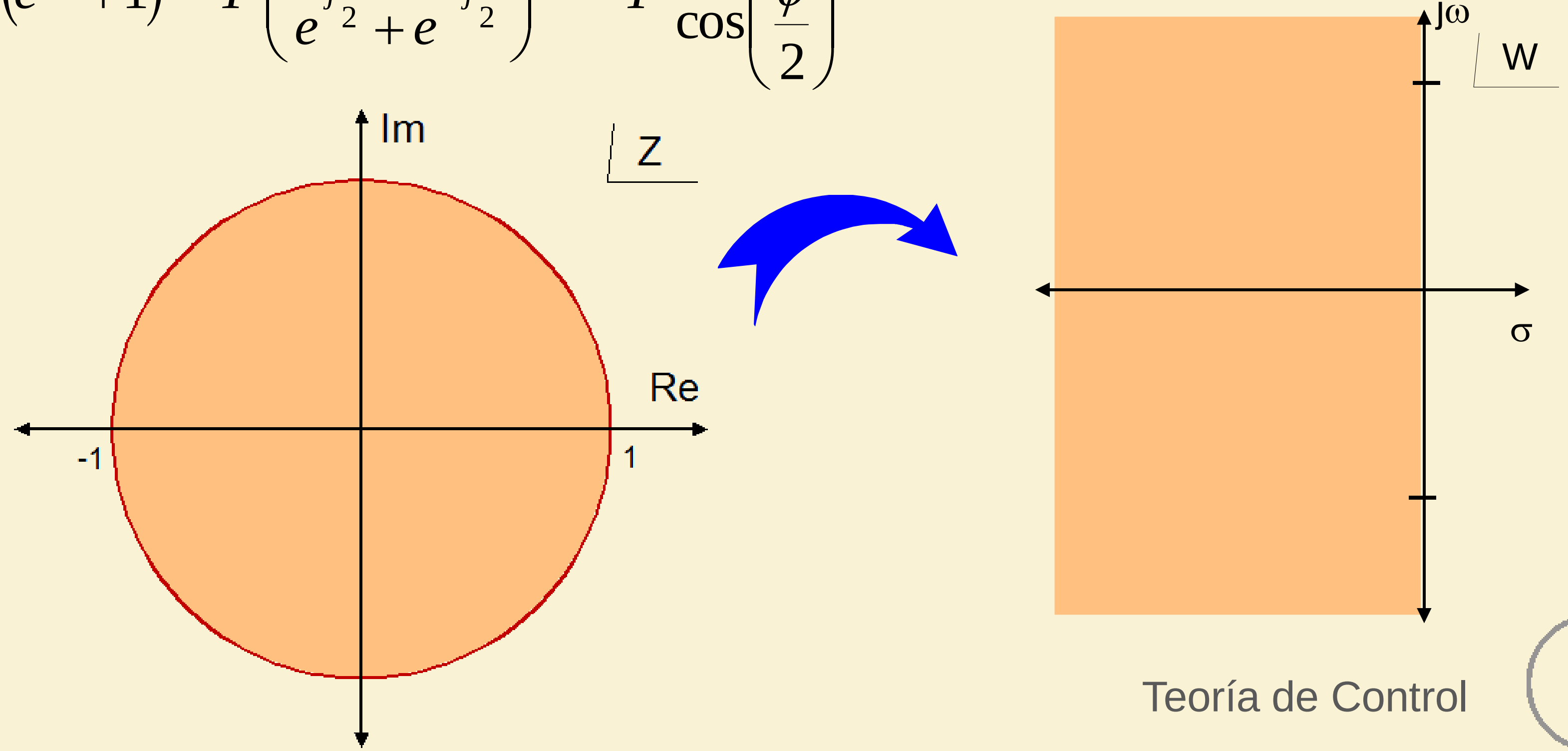
SISTEMAS DISCRETOS

TRANSFORMACIÓN BILINEAL

$$w = \frac{2}{T}\frac{(z-1)}{(z+1)} \qquad z = \frac{\left(1+\frac{wT}{2}\right)}{\left(1-\frac{wT}{2}\right)}$$

$$z = 1_{\angle\phi} = e^{j\phi}$$

$$w = \frac{2}{T}\frac{\left(e^{j\phi}-1\right)}{\left(e^{j\phi}+1\right)} = \frac{2}{T}\left(\frac{e^{j\frac{\phi}{2}}-e^{-j\frac{\phi}{2}}}{e^{j\frac{\phi}{2}}+e^{-j\frac{\phi}{2}}}\right) = j\frac{2}{T}\frac{sen\left(\frac{\phi}{2}\right)}{\cos\left(\frac{\phi}{2}\right)}$$

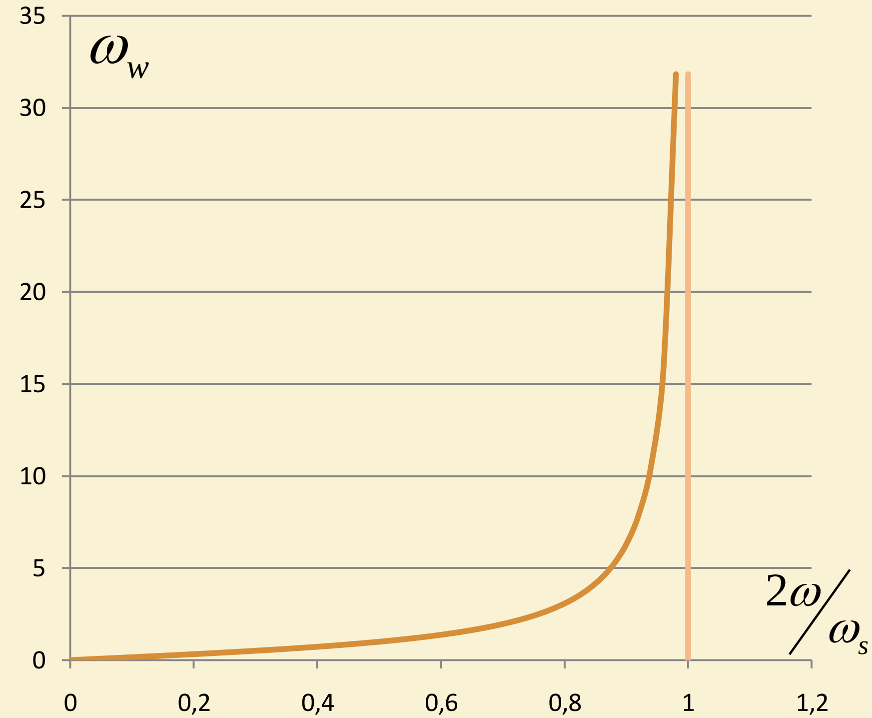


TRANSFORMACIÓN BILINEAL

$$z = e^{sT}\Big|_{s=j\omega} = e^{j\omega T}$$

$$w = \frac{2}{T}\frac{(z-1)}{(z+1)}$$

$$w = j\frac{2}{T}\frac{sen\left(\frac{\omega T}{2}\right)}{\cos\left(\frac{\omega T}{2}\right)} = j\frac{2}{T}\tan\left(\frac{\omega T}{2}\right)$$



si $\left(\frac{\omega T}{2}\right) << 1 \qquad w = \sigma_w + j\omega_w \approx j\frac{2}{T}\left(\frac{\omega T}{2}\right)$

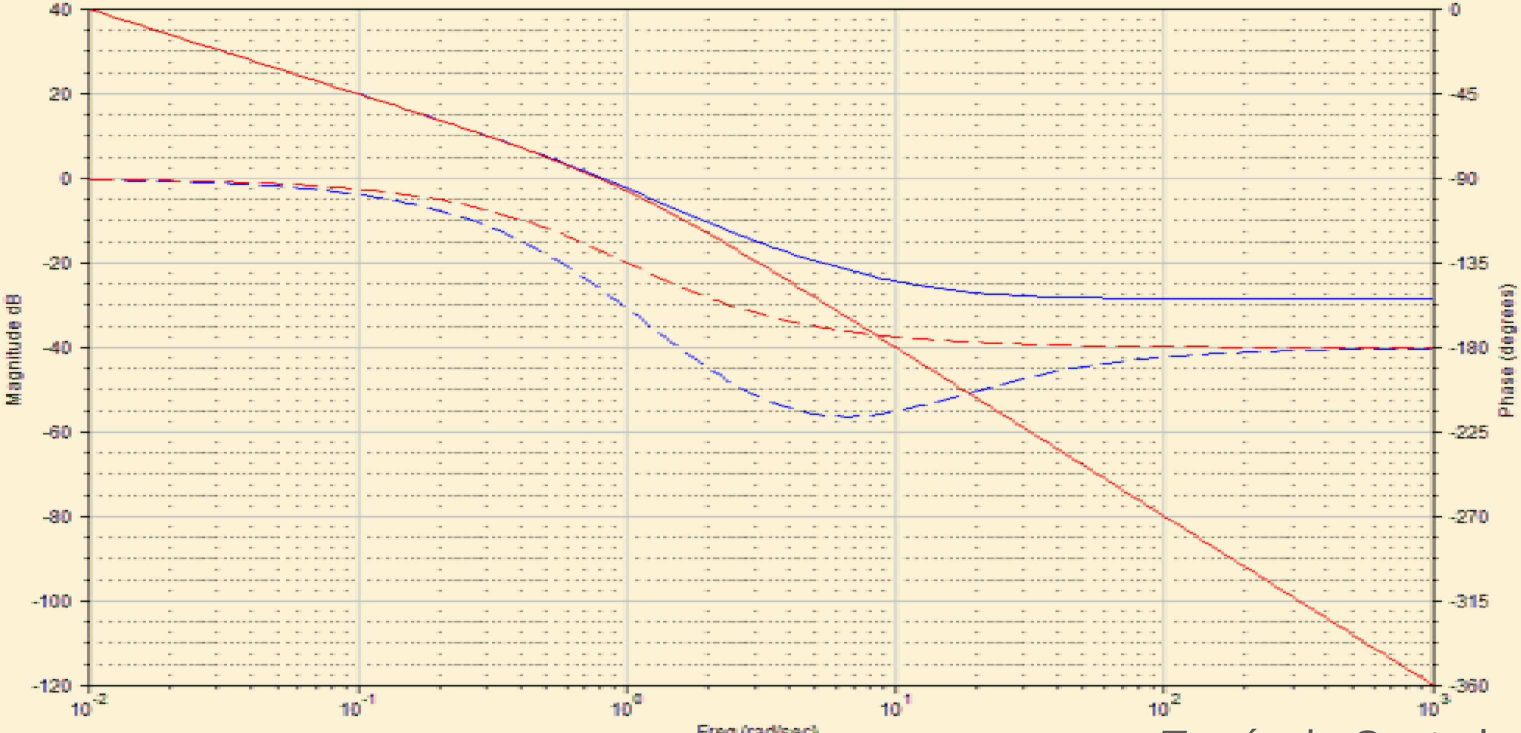
$$\omega_w \approx \omega$$

SISTEMAS DISCRETOS

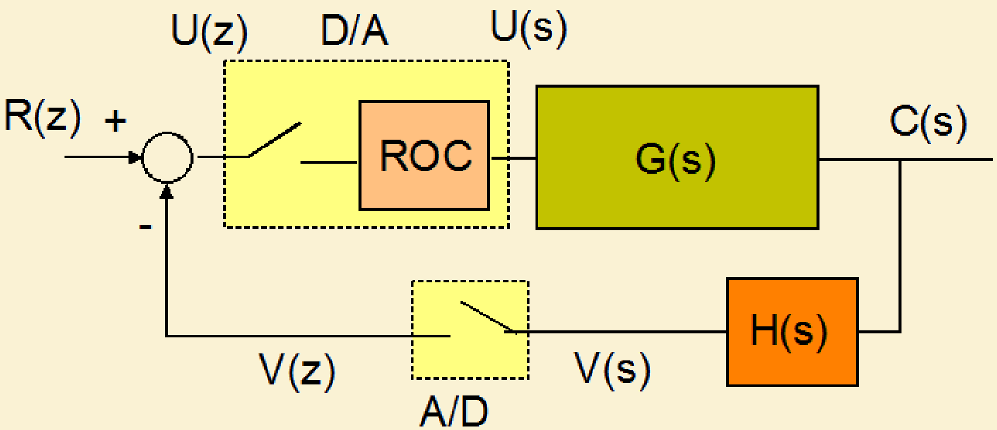
TRANSFORMACIÓN BILINEAL

$$G(s)=\frac{1}{s(s+1)} \quad H(s)=1 \quad T=1 \qquad G(z)=\frac{0,3679(z+0,7183)}{(z-1)(z-0,3679)}$$

$$G(w)=\frac{-0,03788(w+12,2)(w-2)}{w(w+0,9242)}$$



APROXIMACIÓN DE LA RETENCIÓN DE ORDEN CERO



$$G_{ROC}(j\omega) = T\frac{sen\left(\omega\pi/\omega_s\right)}{\left(\omega\pi/\omega_s\right)}e^{-j\left(\omega\pi/\omega_s\right)}$$

$$G_{ROC}(j\omega)GH^*(j\omega) = \frac{1}{T}\sum_{-\infty}^{\infty} G_{ROC}(j\omega + jn\omega_s)GH(j\omega + jn\omega_s)$$

$$G_{ROC}(j\omega)GH^*(j\omega) = \frac{1}{T}\sum_{-\infty}^{\infty} T\frac{sen\left(\omega+n\omega_s/2\right)}{\left(\omega+n\omega_s/2\right)}e^{-j\left(\omega+n\omega_s/2\right)T}GH(j\omega + jn\omega_s)$$

$$G_{ROC}(j\omega)GH^*(j\omega) = GH(j\omega)e^{-j\omega T/2}$$

APROXIMACIÓN DE LA RETENCIÓN DE ORDEN CERO