8.8 A State Observer of Reduced Order

In the identity observer of section 8.6 all state variables $\underline{x}(k)$ are reconstructed. However, if some state variables are directly measurable they need not be calculated. For example, in an m^{th} order process with one input and one output one state variable can directly be calculated from the measurable output $y(k)$, so that only $(m-1)$ state variables have to be determined by the observer. An observer whose order is lower than the order of the process model is called an *observer of reduced order* (see [8.13], [8.15]). The following derives a reduced observer using [8.15] and [2.19]. The process is assumed to be

$$\underline{x}(k+1) = \underline{A}\,\underline{x}(k) + \underline{B}\,\underline{u}(k) \qquad (8.8\text{-}1)$$

$$\underline{y}(k) = \underline{C}\,\underline{x}(k). \qquad (8.8\text{-}2)$$

The dimensions of the vectors are

$\underline{x}(k)$: (mx1)
$\underline{u}(k)$: (px1)
$\underline{y}(k)$: (rx1).

For r independent measurable output variables $\underline{y}(k)$, r state variables can be calculated directly. Therefore, the state vector $\underline{x}(k)$ is partitioned into a directly calculable part $\underline{x}_b(k)$ and an observable part $\underline{x}_a(k)$:

$$\begin{bmatrix} \underline{x}_a(k+1) \\ \underline{x}_b(k+1) \end{bmatrix} = \begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{bmatrix} \begin{bmatrix} \underline{x}_a(k) \\ \underline{x}_b(k) \end{bmatrix} + \begin{bmatrix} \underline{B}_1 \\ \underline{B}_2 \end{bmatrix} \underline{u}(k) \qquad (8.8\text{-}3)$$

$$\underline{y}(k) = [\underline{C}_1 \ \underline{C}_2] \begin{bmatrix} \underline{x}_a(k) \\ \underline{x}_b(k) \end{bmatrix}. \tag{8.8-4}$$

The directly calculable state vector $\underline{x}_b(k)$ is replaced by $\underline{y}(k)$. Then, a state vector $\underline{v}$ is obtained by the linear transformation:

$$\underline{v} = \begin{bmatrix} \underline{x}_a \\ \underline{y} \end{bmatrix} = \underline{T}\ \underline{x} = \begin{bmatrix} \underline{T}_{11} & \underline{T}_{12} \\ \underline{T}_{21} & \underline{T}_{22} \end{bmatrix} \begin{bmatrix} \underline{x}_a \\ \underline{x}_b \end{bmatrix}. \tag{8.8-5}$$

It follows from Eq. (8.8-4) that $\underline{T}_{21} = \underline{C}_1$ and $\underline{T}_{22} = \underline{C}_2$. As $\underline{x}_a(k)$ remains unchanged, $\underline{T}_{11} = \underline{I}$, and is independent of $\underline{x}_b(k)$, $\underline{T}_{12} = \underline{O}$. Therefore, the transformation matrix is

$$\underline{T} = \begin{bmatrix} \underline{I} & \underline{O} \\ \underline{C}_1 & \underline{C}_2 \end{bmatrix} \tag{8.8-6}$$

and the transformed process is

$$\underline{v}(k+1) = \underline{A}_t \underline{v}(k) + \underline{B}_t \underline{u}(k) \tag{8.8-7}$$

$$y(k) = \underline{C}_t \underline{v}(k). \tag{8.8-8}$$

Hence it follows from Eq. (3.2-32) that

$$\left.\begin{aligned} \underline{A}_t &= \underline{T}\ \underline{A}\ \underline{T}^{-1} \\ \underline{B}_t &= \underline{T}\ \underline{B} \\ \underline{C}_t &= \underline{C}\ \underline{T}^{-1} = [\underline{O}\ \underline{I}]. \end{aligned}\right\} \tag{8.8-9}$$

If Eq. (8.8-7) is partitioned as in Eq. (8.8-3) we obtain

$$\underline{x}_a(k+1) = \underline{A}_{t11}\ \underline{x}_a(k) + \underline{A}_{t12}\ \underline{y}(k) + \underline{B}_{t1}\ \underline{u}(k) \tag{8.8-10}$$

$$\underline{y}(k+1) = \underline{A}_{t21}\ \underline{x}_a(k) + \underline{A}_{t22}\ \underline{y}(k) + \underline{B}_{t2}\ \underline{u}(k). \tag{8.8-11}$$

In Eq. (8.8-10) an identity observer of order (m-r) is now used

$$\hat{\underline{x}}_a(k+1) = \underline{A}_{t11}\ \hat{\underline{x}}_a(k) + \underline{A}_{t12}\ \underline{y}(k) + \underline{B}_{t1}\ \underline{u}(k) + \underline{H}\ \underline{e}_t(k) \tag{8.8-12}$$

(c.f. Eq. (8.6-3)). With the identity observer of complete order m the output error given by Eq. (8.6-2) is used for the error between the observer and the process. However, as the reduced order observer does not explicitly calculate $\hat{\underline{y}}(k)$, and as $y(k)$ contains no information concerning $\hat{\underline{x}}_a(k)$, the observer error $\underline{e}_t(k)$ must be redefined. Here, Eq. (8.8-11) can be used because it yields an equation error $\underline{e}_t(k)$ if $\hat{\underline{x}}_a(k)$ is not yet adapted to the measurable variables $\underline{y}(k)$, $\underline{y}(k+1)$ and $\underline{u}(k)$

$$\underline{e}_t(k) = \underline{y}(k+1) - \underbrace{\underline{A}_{t21}\,\hat{\underline{x}}_a(k) - \underline{A}_{t22}\,\underline{y}(k) - \underline{B}_{t2}}_{\hat{\underline{y}}(k+1)}\,\underline{u}(k). \quad (8.8\text{-}13)$$

From Eq. (8.8-12) and Eq. (8.8-13) the observer becomes

$$\begin{aligned}\hat{\underline{x}}_a(k+1) &= \underline{A}_{t11}\,\hat{\underline{x}}_a(k) + \underline{A}_{t12}\,\underline{y}(k) + \underline{B}_{t1}\,\underline{u}(k)\\ &\quad + \underline{H}[\underline{y}(k+1) - \underline{A}_{t22}\,\underline{y}(k) - \underline{B}_{t2}\,\underline{u}(k) - \underline{A}_{t21}\,\hat{\underline{x}}_a(k)].\end{aligned} \quad (8.8\text{-}14)$$

Its block diagram is shown in Fig. 8.8.1. In Eq. (8.8-14), $\underline{y}(k+1)$ is unknown at time k. Fig. 8.8.1 shows that, with respect to the output $\hat{\underline{x}}_a(k)$, nothing changes if the signal path

$$\hat{\underline{x}}_a(z) = \underline{H}\,z^{-1}\,\underline{y}(z)\,z$$

is replaced by

$$\hat{\underline{x}}_a(z) = \underline{H}\,\underline{y}(z).$$

However, we must introduce new observer state variables

$$\hat{\underline{\mu}}(k) = \hat{\underline{x}}_a(k) - \underline{H}\,\underline{y}(k). \quad (8.8\text{-}15)$$

From Fig. 8.8.2 the observer of reduced order is

$$\begin{aligned}\hat{\underline{\mu}}(k+1) &= \underline{A}_{t11}\,\hat{\underline{\mu}}(k) + [\underline{A}_{t12} + \underline{A}_{t11}\,\underline{H}]\,\underline{y}(k) + \underline{B}_{t1}\,\underline{u}(k)\\ &\quad + \underline{H}[-\underline{A}_{t21}\,\underline{H}\,\underline{y}(k) - \underline{A}_{t22}\,\underline{y}(k) - \underline{B}_{t2}\,\underline{u}(k) - \underline{A}_{t21}\,\hat{\underline{\mu}}(k)]\end{aligned}$$

or

$$\begin{aligned}\hat{\underline{\mu}}(k+1) &= [\underline{A}_{t11} - \underline{H}\,\underline{A}_{t21}]\,\hat{\underline{\mu}}(k)\\ &\quad + [\underline{A}_{t12} - \underline{H}\,\underline{A}_{t22} + \underline{A}_{t11}\,\underline{H} - \underline{H}\,\underline{A}_{t21}\,\underline{H}]\,\underline{y}(k)\\ &\quad + [\underline{B}_{t1} - \underline{H}\,\underline{B}_{t2}]\,\underline{u}(k).\end{aligned} \quad (8.8\text{-}16)$$

The state variables to be observed are obtained from

$$\hat{\underline{x}}_a(k) = \hat{\underline{\mu}}(k) + \underline{H}\,\underline{y}(k) \quad (8.8\text{-}17)$$

and finally the overall state vector is

$$\hat{\underline{v}}(k) = \begin{bmatrix}\hat{\underline{x}}_a(k)\\ \underline{y}(k)\end{bmatrix} = \begin{bmatrix}\underline{I} & \underline{H}\\ \underline{0} & \underline{I}\end{bmatrix}\begin{bmatrix}\hat{\underline{\mu}}(k)\\ \underline{y}(k)\end{bmatrix}. \quad (8.8\text{-}18)$$

Considering the state variable error of the reduced observer

$$\tilde{\underline{x}}_a(k+1) = \underline{x}_a(k+1) - \hat{\underline{x}}_a(k+1) \quad (8.8\text{-}19)$$

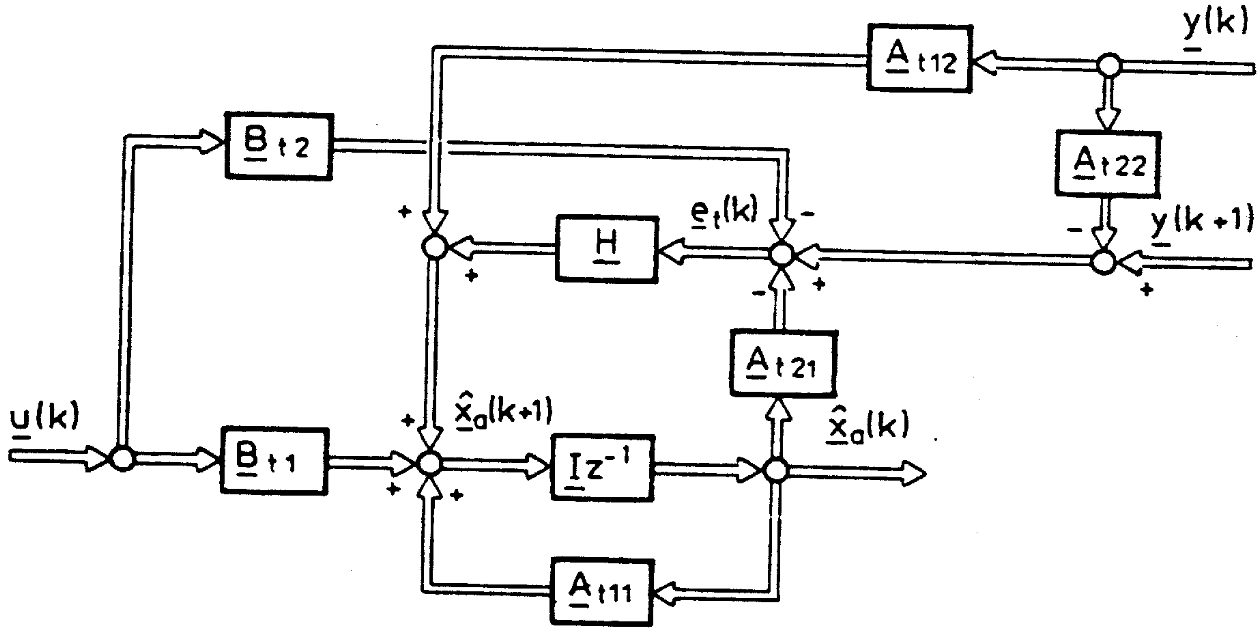

Figure 8.8.1 Block diagram of a reduced-order observer given by Eq. (8.8-14)

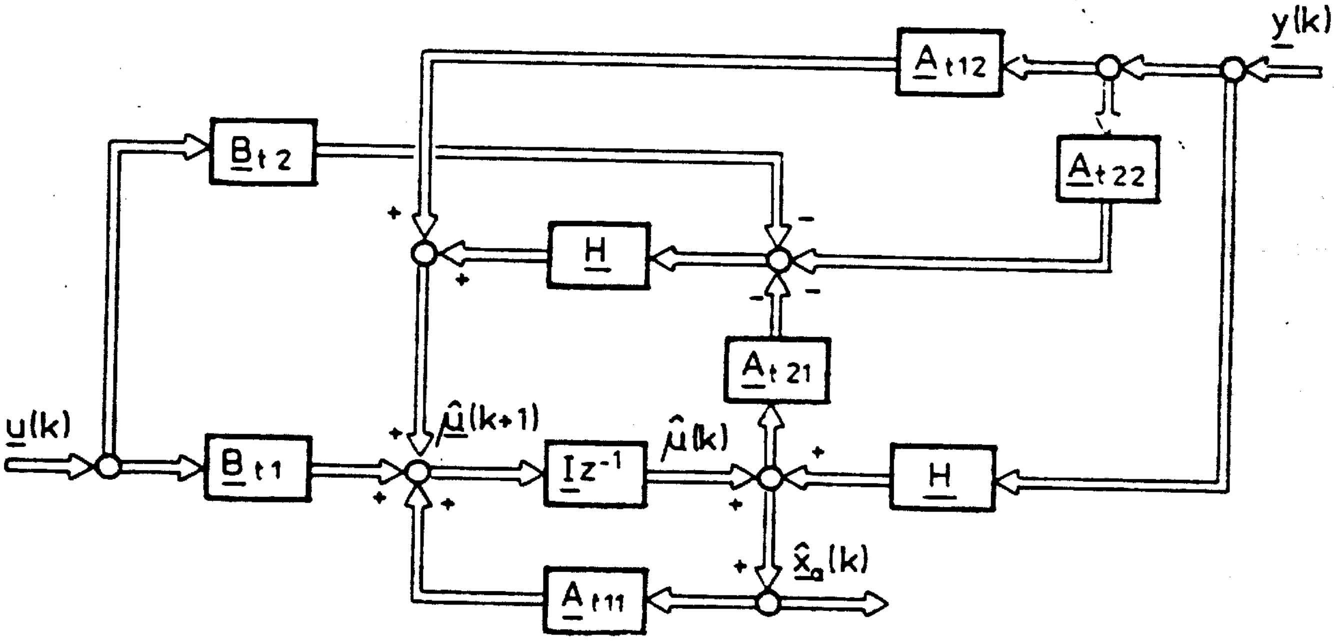

Figure 8.8.2 Modified block diagram of a reduced-order observer given by Eq. (8.8-15)

it follows from Eq. (8.8-10) to (8.8-13) that

$$\underline{\tilde{x}}_a(k+1) = [\underline{A}_{t11} - \underline{H}\,\underline{A}_{t21}]\,\underline{\tilde{x}}_a(k). \qquad (8.8\text{-}20)$$

Compared with the identity observer, in this homogeneous error difference equation one takes the transformed part system matrix $\underline{A}_{t11}$ which belongs to the state vector $\underline{x}_a$, rather than the system matrix $\underline{A}$, and instead of the output matrix $\underline{C}$ one takes the transformed part system matrix $\underline{A}_{t21}$, yielding the relationship between $\underline{x}_a(k)$ and $\underline{y}(k+1)$ given by Eq. (8.8-11).

The characteristic equation of the reduced observer is

$$\det [z\,\underline{I} - \underline{A}_{t11} + \underline{H}\,\underline{A}_{t21}] = (z-z_1)\,(z-z_2)\,\ldots\,(z-z_{m-r}) = 0. \qquad (8.8\text{-}21)$$

The observer poles can be determined using section 8.6.

The advantages of a reduced-order observer compared with the identity observer of section 8.6 are in its lower order (reduced by the number r of the directly measurable output variables) and in the use of current output variables $\underline{y}(k)$ with no delay, hence avoiding the delays described in section 8.7. These advantages, however, are offset by the increased computations required. Moreover, an additional equation, Eq. (8.8-17), arises in the calculation of the state variables to be observed. In digital computer realization, a reduced-order observer is usually preferred if relatively many state variables are directly measurable. In all other cases, e.g. for processes with only one measurable input and output, the identity observer modified according to section 8.7 is better as the design is simpler and more transparent and the potential saving of operational calculations is comparatively small.