Bifurcations and Chaos in the Tolerance Band PWM Technique
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Abstract—In this paper we study the dynamical behavior of the tolerance band PWM technique, which is used in controlling power electronic circuits. We demonstrate numerically as well as experimentally the existence of three basic modes: quasiperiodic, chaotic, and square wave mode. We also observe saddle node bifurcation as the cause of boundary crises with transient chaos, merging crises following symmetry breaking bifurcations and interior crisis. The critical value of the amplitude of external reference for saddle node bifurcation is evaluated analytically by the Tsypkin method.

Index Terms—Chaos, hysteresis band method, PWM, tolerance band method.

I. INTRODUCTION

Most power electronic circuits are controlled by pulse width modulation (PWM) schemes. Various PWM control schemes are now available, each suitable for particular applications. A systematic categorization can be found in [1].

It has been reported that dc–dc converters with the PWM principles of current mode control [2], [3] and duty cycle control [4]–[9] exhibit chaotic behavior over a wide range of parameter values. Various nonlinear phenomena and pathways to chaos have been investigated.

In this paper we investigate the dynamics of another PWM technique, the tolerance band method [1]. This PWM technique is generally applied to current controlled ac inverters for variable frequency drives and uninterruptible power supply (UPS) systems. In addition, modern concepts for voltage and current control in three-phase ac inverters with decoupling of state space variables include this method [10]. Therefore, the nonlinear dynamics of the tolerance band PWM technique is of importance to the engineering community.

For the tolerance band method, no time base exists. The switching action is controlled by the upper and lower threshold voltages of an on–off controller with hysteresis. One of the system variables is compared with the tolerance band around the reference waveform. If the state variable tends to go above or below the tolerance band, the appropriate switching action takes place and the variable is forced to follow the reference waveform within the tolerance band.

Depending on the control strategy the feedback can be generated by the load-voltage, the load-current or the inverter output current. We study the load-voltage feedback with resistive load \( R \), as shown in Fig. 1. The variable frequency is produced by variation of the frequency of the sinusoidal reference signal \( W(\tau) \). To eliminate the higher harmonics of the load current the LC filter is used.

Dc–dc converters can also use this principle, where the sinusoidal reference is changed to a constant value.

II. MODEL OF THE SYSTEM, DIFFERENTIAL EQUATION, AND THE POINCARÉ MAP

From the point of view of control engineering the system in Fig. 1 shows a nonautonomous relay connected to a linear plant of second order. Because the switching characteristic of the controller and the linearity of the plant, the nonlinearity can be reduced to piecewise linearity. This allows analytic integration of the differential equation in each piece. The dependence of the signals on time is described by the normalized time variable \( \tau \), given by \( \tau = \omega_0 t \), where \( \omega_0 = \left( \sqrt{LC} \right)^{-1} \) is the resonant frequency of the linear system of second order. Also, the frequency of the sinusoidal drive signal \( \omega_0 \) is normalized to \( \Omega \) when the time \( t \) is substituted, i.e., \( \omega_0 t = (\omega_0/\omega_0) \tau = \Omega \tau \). Furthermore, the correcting variable of the controller \( u(\tau) \), the output voltage \( z(\tau) \), and the drive signal \( w(\tau) \) and the corresponding amplitude \( \sigma \), the error signal \( e(\tau) \), and the hysteresis or the width of the tolerance band of the controller \( h \) are normalized to the maximum value of the on–off controller output voltage \( u_m \).

\[
U(\tau) = \frac{u(\tau)}{2u_m} = \frac{1}{2} \quad X(\tau) = \frac{x(\tau)}{2u_m} \\
W(\tau) = \frac{w(\tau)}{2u_m} \quad E(\tau) = \frac{e(\tau)}{2u_m} \\
A = \frac{a}{2u_m} \quad H = \frac{h}{2u_m}.
\]

In normalized notation, the expression 2 \( H \) is the width of the tolerance band. \( H \) represents the threshold value where the switching action takes place. The piecewise linear differential equation in normalized notation is as follows:

\[
X''(\tau) + 2DX'(\tau) + X(\tau) = U(E(\tau))
\]

where

\[
E(\tau) = W(\tau) - X(\tau) \\
W(\tau) = A \cos(\Omega \tau + \Phi)
\]
The value of the damper $D = (1/2R)\sqrt{L/C}$, which follows from the linear passive components $R$, $L$ and $C$ in Fig. 1 is chosen less than one in practical circuits [10]. This gives a complex conjugate pole pair in the transfer function of the linear system, implying damped oscillations in the time domain. Amplitude $A$ and frequency $\Omega$ are the parameters of excitation. While frequency $\Omega$ is varied for studying the system, the other parameter values are kept fixed at $A = 0.4$, $D = 0.25$ and $H = 0.005$ in the following analysis. Moreover, the differential equation is formulated by use of the error signal $E(\tau)$ instead of $X(\tau)$, which effects no qualitative difference but the equation is handsomer. If $E(\tau)$ is chaotic, $X(\tau)$ is also chaotic because the difference between the signals is just the sinusoidal reference function $W(\tau)$. To simplify the model we define new variables

$$E_1 = E(\tau), \quad E_2 = E'(\tau), \quad E_3 = \Omega \tau + \Phi,$$

(4)

$E_3$ in this nonautonomous system characterizes the dependence on time. Using the appropriate substitution (4) the nonautonomous differential equation (2) can be written as three equations of first order as follows:

$$
\begin{align*}
E_1' &= E_2 \\
E_2' &= -U(E_1) - 2DE_2 - E_1 + AG(\Omega)^{-1} \cos(E_3 - \Psi_G(\Omega)) \\
E_3' &= \Omega.
\end{align*}
$$

(5)

Let $G(\Omega)^{-1}$ be the absolute value of the inverted transfer function and $\Psi_G(\Omega)$ the corresponding phase displacement of the linear system

$$
G(\Omega)^{-1} = \sqrt{1 - \Omega^2} + i(2D\Omega^2)
$$

(6)

Equation (5) is linear until $U(E_1)$ changes its value. Because of the relay characteristic (3) only two linear regions are obtained. The trajectory in state space changes at the crossing of the threshold from one linear region to the other. The complete solution of the equation must be assembled from two parts which alter from switching point to switching point. The final condition of one switching phase is the initial condition of the next phase. The solution for the linear regions reads as follows:

$$
\begin{align*}
E_1(\tau) &= -\frac{1}{2}(-1)^k + A \cos(2\tau + E_{3k}) \\
&\quad + e^{-(\Omega \tau + \Phi)} \cos(\sqrt{D^2 - 1}\tau) \\
&\quad \cdot \left( E_{1k} + \frac{1}{2}(-1)^k - A \cos E_{3k} \right) \\
&\quad + e^{-(\Omega \tau + \Phi)} \sin(\sqrt{D^2 - 1}\tau) \\
&\quad \cdot \left( E_{2k} + D E_{1k} + D \frac{1}{2}(-1)^k - A(D \cos E_{3k} - \Omega \sin E_{3k}) \right) \\
&\quad = f(E_{1k}, E_{2k}, E_{3k}, \tau).
\end{align*}
$$

(7)

Equation (7) represents the condition for the $(k+1)$th switching instant, where the threshold $H$ is reached and the trajectory changes from one linear region to the other. The time $\tau_n$ between the switching instants must be evaluated numerically to find the solution of the transcendental equation. The computation is done using the Anderson–Bjoerk method for high convergence rate of the numerical process. Thereafter, the system variables at the $(k+1)$th instant can be calculated where $g$ is the derivation of time of function $f$ (6). To simplify the generation of Poincaré maps using the experimental circuit, the discrete observations were made at the zero crossing of the drive signal $W(\tau)$ with positive slope instead of the switching instant of the controller. To facilitate the comparison of results, the numerical evaluation of the Poincaré map was done in the same way. In this representation, the discrete state variables at the $n$th zero crossing are $E_{1n}$ and $E_{2n}$. The number $n$ of zero crossings and $k$ of switching instants are independent of each other. Within one period of $W(\tau)$ several switching instants may take place depending on the oscillation mode of the system. Fig. 2 shows this situation by a timing chart of the signals $E(\tau)$ and $U(\tau)$ of a regular PWM mode. The time $\tau$ here is scaled continuous and does not start at each switching instant. If the condition $E_{3(k+1)} > E_{3n} = 2\pi n - \pi/2 > E_{3k}$ is satisfied within the actual switching phase, a zero crossing of the drive signal will occur. Therefore, a Poincaré section must be obtained using the following algorithm:

$$
\begin{align*}
\Omega \tau_{n+1} &= 2\pi n - \frac{\pi}{2} - E_{3k} \\
E_{1(n+1)} &= f(E_{1k}, E_{2k}, E_{3k}, \tau_n) \\
E_{2(n+1)} &= g(E_{1k}, E_{2k}, E_{3k}, \tau_n).
\end{align*}
$$

(8)

For obtaining experimental Poincaré maps an electronic circuit has been built. The controller was implemented using a 74HC08 AND gate as the comparator, as shown in Fig. 3. Hysteresis $h$ of the controller follows from the positive feedback of the voltage divider resistors 100 k$\Omega$ and 1 k$\Omega$ the linear system was implemented by a $LCR$-resonant circuit. $(u_m = 2.5$ V, $h = 50$ mV, $L = 0.185$ H, $C = 1.36$ nF, $R = 24$ k$\Omega$). Setting the right value of the damper ($D = 0.25$), the resonant circuit requires a slightly higher value of the resistor $R$ to account for the iron losses in the inductor $L$. The drive signal with variable frequency was obtained from an audio sine wave generator ($f_o = (\omega_o/2\pi) \approx [13$ kHz, $15$ kHz] and $a = 2$ V). The reduction of the frequency to lower values (e.g., $60$ Hz), as used in some practical applications [10], is possible without loss of validity. Only the values of the components of the $LCR$-resonant circuit must be changed.
III. GLOBAL BEHAVIOR AND BIFURCATIONS

At low-frequency operation as in Fig. 2, the signal \( U(\tau) \) shows a PWM mode where the signal \( E(\tau) \) is quasiperiodic. Fig. 4 shows the corresponding experimentally obtained Poincaré section. The scaling factors of Fig. 4 for both channels correspond to the values before normalization \( \Omega = 0.3, A = 0.4, D = 0.25, \) and \( H = 0.005 \).

We observe a folding of the torus as \( \Omega \) is increased. Fig. 4(b) shows the experimentally obtained Poincaré section at \( \Omega = 1.399 \) in chaotic mode. The chaotic behavior is confirmed by a positive maximal Lyapunov exponent \( \lambda_1 = +0.234 \). It has been calculated numerically from the discrete Poincaré map at the switching point.

The phenomena in the intermediate region are shown in the numerically obtained bifurcation diagram given in Fig. 5. In the diagram, \( E_{1,n} \) represents one component of the Poincaré map at the zero crossing of the driving signal with positive slope. The critical values of the parameter \( \Omega \) where important bifurcation phenomena occur are marked in the figure. In this system we observe coexisting attractors \( a_1 \) and \( a_2 \) and separate bifurcation diagrams are given in Fig. 5(a) and (b) to illustrate the evolution of these attractors.

A. Symmetry Breaking

In a sinusoidally forced system (5), if the Poincaré map at the observation angle \( \Psi_{\alpha_0} = E_{3,n} = 2n \pi - (\pi/2) \) is in rotation symmetry with the map at \( \Psi_{\alpha_\pi} = \Psi_{\alpha_0} + \pi \), we call the attractor invariant under the symmetry group \( \Gamma \). In this case, the Poincaré section at \( \Psi_{\alpha_\pi} \), under a change of sign (i.e., \( -E_{1,n}, -E_{2,n} \)), coincides with the section at angle \( \Psi_{\alpha_0} \). In other words, the map at \( \Psi_{\alpha_0} \) coincides with the map at \( \Psi_{\alpha_\pi} \), rotated by 180°.

In the system under consideration there is no preferred direction of deflection of the system variables as the driving term, the nonlinear relay characteristic, and the linear part are centred on the origin. Therefore, the occurrence of attractors invariant under the group \( \Gamma \) can be expected.

It can be observed in the bifurcation diagrams of Fig. 5(a) and (b) that at some bifurcation points the attractor is replaced by a pair of
coexisting attractors $a_1$ and $a_2$. Both attractors of the bifurcated system have their own basins of attraction. They are of reduced symmetry and only the set $a_1 \cup a_2$ is symmetric under the group $\Gamma$. This phenomenon is called symmetry breaking. An explanation of the symmetry breaking phenomenon can be found in [13]. Such paired solutions have also been observed in the Duffing and the Lorenz systems [14].

Fig. 6 shows the Poincaré section at the parameter value $\Omega = 1.387$ where a pair of coexisting chaotic attractors $a_1$ and $a_2$ can be seen. Fig. 6(a) shows $a_1$ and $a_2$ when observed at $\Psi_{\pi}$ and Fig. 6(b) shows the same attractors at $\Psi_{2\pi}$. The attractors $a_1$ and $a_2$ are not invariant under the symmetry group $\Gamma$, they do not coincide with themselves by rotating one of the maps by $180^\circ$. However, the attractor $a_2$ at $\Psi_{2\pi}$ coincides with the attractor $a_1$ at $\Psi_{\pi}$ by a rotation of $180^\circ$. These are the less symmetric attractors. Only in the set $a_1 \cup a_2$ a rotation of $180^\circ$ will coincide and the symmetry under the group of $\Gamma$ is guaranteed.

As the frequency is increased, the first symmetry breaking occurs at $\Omega_{CRA_{1 \& 2}}$ and a pair of less symmetric attractors of the same periodicity are born. These attractors undergo period doubling bifurcations as $\Omega$ is increased and become chaotic.

Another symmetry breaking phenomenon occurs at a saddle node bifurcation at frequency $\Omega_{CRA_{1 \& 2}}$. As the frequency decreases and passes this critical point from above, two period-one attractors are born. The attractors undergo a period doubling cascade until $\Omega_{CRA_{1 \& 2}}$.

### B. Merging Crisis

At $\Omega_{CRA_{1 \& 2}}$ and $\Omega_{CRA_{1 \& 2}}$, the two attractors merge and there is a sudden expansion of the chaotic attractor. This is a merging crisis [15]. In the range $\Omega > \Omega_{CRA_{1 \& 2}}$, the two attractors coexist and at $\Omega = \Omega_{CRA_{1 \& 2}}$, both of them collide with their own basin boundaries. For $\Omega$ slightly smaller than $\Omega_{CRA_{1 \& 2}}$, an orbit typically spends long stretches of time moving chaotically in the region of one of the old attractors, after which it abruptly switches to the region of the other attractor, intermittently switching between the two. Fig. 7 shows the merged attractor at frequency $\Omega = 1.386$. The long stretches can be recognized by the accumulation of data points at the region of the old attractors $a_1$ and $a_2$. The above mentioned considerations are also valid for $\Omega_{CRA_{1 \& 2}}$ by increasing the frequency.
collide with the chaotic attractor at frequency $\Omega_{CR2}$, which results in an interior crisis.

D. Saddle Node Bifurcation

The square wave mode is an effect caused by a saddle node bifurcation. An exact critical amplitude bound $A_{CR}$ can be evaluated analytically, at which a saddle node bifurcation takes place. Above this bound, a pair of stable and unstable symmetrical limit cycles with frequency of the drive signal $\omega(\tau)$ exists and the external forcing is synchronized to the system.

At this synchronized steady state, the signal $U(\tau)$ shows a symmetric periodic rectangular function with a duty cycle of 50%. Tsypkin’s method is used for the characterization of the relay system’s limit cycles [16]. The existence of limit cycles is predicted by the fulfillment of algebraic conditions of the functions $E_1(\tau_k)$ and $E_2(\tau_k)$ at the switching point $\tau_k$ (here $\sim$ characterizes periodicity). There are two solutions in respect to the phase angle of the drive signal (4) $E_{AS}(\tau_k) = \pi - E_{AC}(\tau_k)$. Both solutions together are the effect of the saddle node bifurcation and only one of these solutions is stable. Stability of such predicted limit cycles is clarified by a special linearization at the bias point, as suggested by Tsypkin [16]. An application of the simplified Nyquist criterion to sampling systems results in the final inequality for the special case

$$A > A_{CR}$$

$$\frac{D}{\sqrt{1-D^2}} \sin \left( \frac{\sqrt{1-D^2}}{\Omega} \pi \right) - \sinh \left( \frac{D}{\Omega} \pi \right) + \frac{2}{\left( \cosh \left( \frac{D}{\Omega} \pi \right) + \cos \left( \frac{\sqrt{1-D^2}}{\Omega} \pi \right) \right)}$$

This is the Tsypkin bound of the given system. The derived relation for $A_{CR}$ gives an explicit bound for the amplitude. If the amplitude $A_{CR}$ is known and the frequency $\Omega$ is of interest, a numerical solution of (8) must be performed. For the selected critical amplitude $A_{CR} = 0.4$, a frequency $\Omega_{CR1} = 1.3343745$ is obtained for the saddle node bifurcation. A stable attractor $a_3$ is caused by this bifurcation. Note that the saddle node bifurcation takes place at a frequency inside the intermediate region. Therefore, above the critical value $\Omega_{CR1}$ we find a range of frequency where $a_3$ coexists with other attractors discussed in Section III. These can be accessed through variation of initial conditions. This is also true for the chaotic attractor in Fig. 4(b), which has a frequency above the critical value $\Omega_{CR1}$. The attractor $a_3$ is, however, not shown in Fig. 5 since it lies out of range.

E. Boundary Crisis

As the parameter $\Omega$ is raised, the distance between the chaotic attractor and its basin boundary decreases until at a critical $\Omega = \Omega_{CR2}$ value they touch. This is a boundary crisis [15]. At this point, the attractor also touches an unstable periodic orbit that was on the basin boundary before crisis. For $\Omega > \Omega_{CR2}$ the chaotic attractor no longer exists and is replaced by a chaotic transient. In the actual system, the end of the intermediate range at frequency $\Omega_{CR2}$ is a result of the boundary crisis. Above $\Omega_{CR2}$ only the square wave mode exists with a fundamental limit cycle. The numerical value of the critical frequency obtained by simulation is $\Omega_{CR2} \approx 1.40$. Because of the very long time constant of the chaotic transient the value can only be specified approximately.

C. Interior Crisis

A sudden increase in the size of a chaotic attractor occurs when the periodic orbit with which the chaotic attractor collides in the interior of its basin and is called an interior crises [15]. A stable period-three orbit is born by a saddle node bifurcation at $\Omega_{CR5}$. An unstable period-three orbit is also born at this frequency. The unstable branches
In Fig. 8, the situation is shown in the Poincaré surface of section. The chaotic attractor is embedded in its basin of attraction. The saddle fixed point lies on the basin boundary which is built from the stable manifolds. One of the unstable manifolds runs into the chaotic attractor. At the critical value of frequency $\Omega_{c,11}$, the two fixed points are born and diverge with the increase of frequency. Thereby, the unstable saddle is moved in direction of the chaotic attractor. As the saddle fixed point reaches the chaotic attractor at $\Omega = \Omega_{c,12}$, the attractor is destroyed.

IV. CONCLUSIONS

In this paper we have demonstrated the chaotic behavior and the bifurcation scenario of the tolerance band PWM technique numerically as well as experimentally. We have shown that the chaotic behavior in this system results from a folded torus in the state space. If the frequency of the external reference signal is varied, the chaotic behavior can be found in an intermediate range between the regular quasiperiodic PWM mode and the square wave mode of the controller. If chaos is to be avoided for certain industrial applications, this intermediate range of frequency specifies the forbidden zone in the parameter space.

With the analytically evaluated Tsypkin bound, the frequency range can be calculated in which a regular PWM operation is reliable. The choice of control strategy (type of feedback), the transfer function $G(\Omega)$ of the linear plant, and the hysteresis $H$ of the controller are the important parameters in the design of such a system.

Furthermore, the coexistence of attractors within the intermediate range has been confirmed, which results in a rich variety of bifurcation phenomena. We have demonstrated the occurrence of saddle node bifurcations, boundary crises, merging crises, interior crises and symmetry breaking bifurcations. We have presented a complete overview of the nonlinear phenomena that can occur in the tolerance band PWM technique: a knowledge necessary for reliable design and operation of such converters and for application of chaos in obtaining flexible switchings.

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