Universal Chaos in Fractional Order Logistic Equation

J.D. Munkhammar

† Studentstaden 23:230, 752 33 Uppsala, Sweden
E-mail: joakim@munkhammar.org

Abstract
Fractional order dynamical systems have attracted increasing interest in recent years. In this paper we investigate a fractional order logistic equation on the basis of fractional calculus. In light of discrete time dynamics we show that the map is unimodal under certain restrictions and numerically that it exhibits chaos as the bifurcation parameter is increased. Furthermore we introduce the concept of studying chaos in fractional dynamical systems with variable fractional parameter under constant bifurcation parameter. As an example of this we hint emerging chaos in the fractional logistic equation under decreased fractional parameter with constant bifurcation parameter.

1 Introduction

The concept of fractional calculus dates back to the genesis of calculus itself since Leibnitz made some remarks on fractional derivative of order 1/2 back in the 17th century. However despite many fundamental results and important definitions found more than 150 years ago by among others Euler and Riemann, the field of applications of fractional calculus has drawn recent interest [2, 5, 6, 7, 10, 14]. Applications to physics involve viscoelastic systems and electro-magnetic waves [5], but more recently dynamics of fractional order dynamical systems, involving fractional mechanics and fractional oscillators has been analyzed [1, 10]. In connection with this new field fractional versions of the Lorenz-system, the Rössler systems [5] and Chen’s system [6] has been investigated. In this paper we investigate the dynamics of a fractional generalization of the well-known logistic equation [15].

2 Dynamics

The dynamics of unimodal maps are well-known and has been extensively studied recently, population dynamics often consider maps like the Ricker-family:

\[ R_{\lambda,\beta}(x) = \lambda x e^{-\beta x}, \quad \lambda > 1 \quad \beta > 0, \quad (2.1) \]

or the Hassel-family (See [15]):

\[ H_{\lambda,\beta}(x) = \frac{\lambda x}{(1 + x)^\beta}, \quad \lambda > 1 \quad \beta > 0, \quad (2.2) \]

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or perhaps more famously the Logistic equation:

\[ Q_\lambda(x) = \lambda x(1 - x), \quad \lambda > 0. \]  

(2.3)

It is defined on the half-line \( x \in [0, \infty[ \), but all of its interesting dynamics takes place on the bounded interval \( I = [0, 1] \) for \( 0 < \lambda \leq 4 \). If one considers discrete time evolution by the mapping \( Q_\lambda : I \to I \), letting the state be \( x_n \) at time \( n \) then one has \( x_{n+1} = Q_\lambda(x_n) \) at time \( n+1 \). In the prescribed interval as the parameter \( \lambda \) is increased the logistic equation experiences a period-doubling route to chaos. For a thorough background on unimodal maps and chaos see [15].

3 Fractional Calculus

There are many models of fractional diff-integration, however the more popular ones are Riemann-Liouville-type operator and Caputo-type operator, whereas the latter one is favored in cases of lack of initial conditions, but in our case they are equivalent.

Definition 1. If \( f(x) \in C([a, b]) \) and \( a < x < b \) then

\[
I^\alpha_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} \, dt 
\]  

(3.1)

where \( \alpha \in ]-\infty, \infty[ \), is called the Riemann-Liouville fractional integral of order \( \alpha \). In the same fashion for \( \alpha \in ]0, \infty[ \) we let

\[
D^\alpha_{a+} f(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} \, dt, 
\]  

(3.2)

which satisfies

\[
D^0_{a+} f(x) = I^0_{a+} f(x) = f(x), 
\]  

(3.3)

is called the Riemann-Liouville fractional derivative of order \( \alpha \).

The fractional derivative comes as a special-case of the fractional integral. A fractional derivative of order \( 1/2 \) is called a semi-derivative and a fractional integral of the same order is called semi-integral. Both the fractional derivative and fractional integral satisfies the important semi-group property:

Theorem 1. For any \( f \in C([a, b]) \) the Riemann-Liouville fractional integral satisfies

\[
I^\alpha_{a+} I^\beta_{a+} f(x) = I^{\alpha+\beta}_{a+} f(x) 
\]  

(3.4)

for \( \alpha > 0, \beta > 0 \).

A proof may be found in [9]. For more information regarding fractional calculus and its applications see [8, 9, 12].
4 Fractional dynamics

4.1 Basic properties of the fractional logistic equation

Many investigations of fractional dynamical systems contain fractional derivatives in time (See [1, 5, 6]), due to their continuous time derivative. As we have discrete time we use the negative order fractional derivative in spatial direction. If we let the Riemann-Liouville fractional integral (3.2) operate on the logistic equation (2.3) we get:

\[ I_{0+}^{\alpha} Q_\lambda(x) = \frac{\lambda}{\Gamma(\alpha + 2)} \left( 1 - \frac{2x}{\alpha + 2} \right) x^{1+\alpha}, \]  

which is valid for at least all \( \alpha \in ] - \infty, \infty[ \) (for more details on fractional differentiation see [8]), thus

**Definition 2.** For all \( x \geq 0, \lambda > 0 \) and \( \alpha \in ] - \infty, \infty[ \):

\[ Q_\lambda^n(x) := \frac{\lambda}{\Gamma(\alpha + 2)} \left( 1 - \frac{2x}{\alpha + 2} \right) x^{1+\alpha}, \]  

which is called the fractional logistic equation (FLE) of order \( \alpha \).

It is left as an exercise to show that the ordinary logistic equation (2.3) follows as a special-case when \( \alpha = 0 \). The Riemann-Liouville fractional operators have special properties for the parameter-value \( \alpha = 1/2 \) and are called semi-integral and semi-derivative respectively (See [8, 9, 12]). Hence we give the definition:

**Definition 3.** For \( x \geq 0 \) and \( \lambda > 0 \) the \( \alpha = 1/2 \) order fractional logistic equation appears like:

\[ Q_{1/2}^\lambda(x) = \frac{4\lambda}{3\sqrt{\pi}} \left( 1 - \frac{4x}{5} \right) x^{3/2}, \]  

and will be called the semi-logistic equation.

Due to the special property of fractional diff-integrals we can obtain the following useful theorem:

**Theorem 2.** For all \( \alpha, n \in ] - \infty, \infty[ \):

\[ \frac{d^n}{dx^n} Q_\lambda^n(x) = Q_{\lambda}^{n-n}(x). \]  

**Proof.** It follows directly from definition 2 and the semi-group property (Theorem 1). \( \blacksquare \)

Although the fractional logistic equation is valid for \( \alpha \in ] - \infty, \infty[ \), the most interesting dynamics takes place as \( \alpha \in [0, 1] \) due to theorem 2 which allows for translation from \( [0, 1] \) to any other domain by \( N \)-times integration/differentiation for some \( N \). When examining discrete maps one tends to look for periodic points and inherent complexity in its dynamics, and in order to look for that we make use of the following theorem [3, p. 140]:

**Theorem 3.** A map \( f : I \to I \) for some \( I = [a, b] \) is unimodal if and only if

1. \( f(a) = f(b) = 0 \).
2. \( f \) has a unique critical point \( c \) with \( a < c < b \).
Unimodality is intimately connected with the chaos (See [3, 13]), as part (1) in Theorem 3 the interval mapping property of the logistic equation is a necessary property of unimodal maps, hence we give the following theorem:

**Theorem 4.** If \( \alpha \in [0, 1] \):

\[
Q_\alpha(0) = Q_\alpha(1 + \frac{1}{2} \alpha) = 0,
\]

holds for all \( \lambda \geq 0 \).

**Proof.** \( Q_\alpha(0) = 0 \) follows by definition. We find \( Q_\alpha(1 + \frac{1}{2} \alpha) = 0 \) by inserting \( x = (1 + \frac{1}{2} \alpha) \):

\[
Q_\alpha(1 + \frac{1}{2} \alpha) = \frac{\lambda}{\Gamma(\alpha + 2)} \left( 1 - \frac{2 + \alpha}{\alpha + 2} \right) \left( 1 + \frac{1}{2} \alpha \right)^{1+\alpha} = 0.
\]

Note that \( \lambda \geq 0 \) is a part of the FLE definition (defn. 2).

Proceeding in the sense of unimodality we look for fixed points, according to the definition of fixed points, the fractional logistic equation must satisfy the condition:

\[
Q_\alpha(x) = x,
\]

which is an equation on the form:

\[
x^{1+\alpha} - \frac{\alpha + 2}{2} x^\alpha + \frac{\Gamma(\alpha + 3)}{2\lambda} = 0.
\]

This equation has non-fractional solutions for \( \alpha \in [1, 2, 3, ...] \) which corresponds to simple integrations of the logistic equation. For the fractional values a general solution is left as an open problem for special-cases of analytic solutions and numerical calculations, as a hint we obtain an analytic solution for a special-case in section 4.2.

### 4.2 Special-case: dynamics of the semi-logistic equation

As denoted before the so-called semi-derivative/integral in fractional calculus has special properties. The corresponding semi-logistic equation has in accordance with (4.8) fixed points when:

\[
x^{\frac{3}{2}} - \frac{5}{4} x^{\frac{1}{2}} + \frac{15\sqrt{\pi}}{16\lambda} = 0
\]

is satisfied. This gives rise to the following theorem:

**Theorem 5.** The semi-logistic equation is unimodal for \( \lambda = 6 \).

**Proof.** We seek solution to the equation:

\[
x^{\frac{3}{2}} - \frac{5}{4} x^{\frac{1}{2}} + \frac{15\sqrt{\pi}}{16\lambda} = 0.
\]

By substituting \( x = z^2 \) we obtain:

\[
z^{3} - \frac{5}{4} z + \frac{15\sqrt{\pi}}{16\lambda} = 0.
\]
A general set of analytic solutions we find by Cardano’s formula (See [16, p. 161] for detailed information regarding Cardano’s formula). Define:

\[ D = -\left(\frac{5}{12}\right)^3 + \left(\frac{15\sqrt{\pi}}{32\lambda}\right)^2, \]  

(4.12)

and we note that there is a critical value as \( D = 0 \) which corresponds to:

\[ \lambda_{\text{crit}} = \frac{15\sqrt{\pi}}{32} \left(\frac{12}{5}\right)^{3/2} \approx 3.089. \]  

(4.13)

According to Cardano’s formula will for all \( \lambda > \lambda_{\text{crit}} \) (Since we consider \( \lambda = 6 \)) have three solutions to (4.11), define:

\[ u = 3 \sqrt[3]{-\frac{15\sqrt{\pi}}{32\lambda} + \sqrt[3]{\left(\frac{5}{12}\right)^3 + \left(\frac{15\sqrt{\pi}}{32\lambda}\right)^2}}, \]  

(4.14)

\[ v = 3 \sqrt[3]{-\frac{15\sqrt{\pi}}{32\lambda} - \sqrt[3]{\left(\frac{5}{12}\right)^3 + \left(\frac{15\sqrt{\pi}}{32\lambda}\right)^2}}, \]  

(4.15)

then the solutions are given by:

\[ z_1 = x_1^2 = u + v \]  

(4.16)

\[ z_2 = x_2^2 = -\frac{u + v}{2} + \frac{u - v}{2}i\sqrt{3} \]  

(4.17)

\[ z_3 = x_3^2 = -\frac{u + v}{2} - \frac{u - v}{2}i\sqrt{3}. \]  

(4.18)

Inserting the values of (4.14) into (4.16) we find that all roots are real, but only \( z_1 \) is positive and relevant because in evaluation of \{ \( x_1, x_2, x_3 \) \} makes \( x_2, x_3 \) complex, we will get the only real root:

\[ x_1 = \sqrt{u + v} \approx 1.10. \]  

(4.19)

Since we have shown existence of uniqueness for a fixed point of the equation, we need to verify that the fixed point lies within the segment \([0,1 + \frac{1}{\alpha}]\). Since \( \alpha = 1/2 \), the segment becomes \([0,3]\) and hence because \( x_1 \approx 1.10 \in [0,3] \) the condition is satisfied. Thus according to theorem 4 the semi-logistic equation for \( \lambda = 6 \) is unimodal.

4.3 Period doubling and chaos

With the aid of Mathematica 5 and software developed in [4] we were able to numerically study the behaviors of the FLE. As an example we plot the FLE in the domain \( x \in [0,1], \) \( \alpha \in [0,1] \):
Note that $x = 1$, $\alpha = 1$ is a local maximum in $x$-direction since the FLE for $x = 1$ and $\alpha = 0$ is zero due to the property of fractional diff-integrals in Theorem 2. Furthermore, as an example if we perform three iterates we end up with:

In fig 4.3 we see $Q_\lambda^\alpha(Q_\lambda^\alpha(Q_\lambda^\alpha(x)))$, which appears to be similar to the logistic map not only for $\alpha = 0$. Indeed, further examinations of the special-case of $\alpha = 1/2$, the semi-logistic equation, we see that a period-doubling route to chaos appears as $\lambda$ is increased from 4.1 to 6.1 (much like the bifurcation-diagram of the logistic equation (See [15])):
Hence a computational verification for chaos indicated by theorem 5.

### 4.4 Fractional Chaos

As a new way of examining the complexity of fractional dynamical systems we let the bifurcation parameter be fixed and vary the fractional parameter. Thus with constant value $\lambda = 1$, we vary $\alpha$ continuously.

The plots in fig 4.4 above indicate as one decrease the fractional parameter $\alpha$ from $0.6$ (Upper left corner) to $0.596$ (Upper right corner) and finally to $0.5$ one sees a clear pattern towards chaos.

### 5 Discussion and open problems

We have shown existence of real fixed points for the FLE under certain restrictions on $\alpha$ and $\lambda$. Furthermore we have shown that the semi-logistic equation is unimodal for $\lambda = 6$. In addition we have numerically shown that chaos emerges for the semi-logistic equation
as the bifurcation parameter is increased and that it follows a period-doubling route to chaos in a similar way to the ordinary logistic equation. We have numerically hinted that chaotic behavior emerges as the fractional parameter is decreased under constant bifurcation-parameter. The analysis as $\alpha$ is varied in fractional dynamical systems has not to the knowledge of the author ever been carried out before. Further investigations of this type on other fractional dynamical systems like the fractional Rössler- and the fractional Lorentz-systems are open issues, and they could possibly contribute to the analysis methods for bifurcation and chaos in fractional-order systems from a frequency domain approach hinted in [5].

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References


