Analyzing Fractal Dimensions and Self-Similarity in Chaotic Light Scattering

Mary Elizabeth Mills

Physics Department, The College of Wooster, Wooster, Ohio 44691, USA

(Dated: May 8, 2008)

The fractal dimension and self-similarity of portions of an image formed by chaotic light scattering were calculated. Chaotic light scattering was achieved by stacking four mirrored spheres into a pyramid and covering three of the four sides with colored cloth; light was shone through the cloth. The fourth, and open, side was used to look into and take pictures of the image formed by the colored light rays. The image produced a fractal and by looking at the center of the fractal, the dimension was calculated to be 1.32 ± 0.04. The dimension of the three fractals on the sides of the image in the center were calculated to be 1.13 ± 0.06. Self-similarity was determined to be present in the fractal.

I. INTRODUCTION

A. Chaotic Light Scattering

Chaotic scattering models complex phenomena in several physical systems. It has multiple applications and has been studied in various areas of physics such as chemical reactions, celestial mechanics, fluid dynamics, atomic and nuclear physics, and electron scattering in semiconductor microstructures [1, 2]. Chaotic scattering with light has been studied by many different people. The majority of the work has come from Dr. Edward Ott at the University of Maryland beginning in the late 1980's. Although most of his work is theoretical and computational, there been some experimental work. David Sweet, a graduate student working under Dr. Ott, investigated the topology of the paths of light rays escaping from the "inner chamber" of four mirrored spheres. The spheres were stacked in a pyramid. Light shone through colored cloth placed over the empty spaces between the spheres [2].

In 2001, David Miller replicated the experimental setup of the University of Maryland Chaos Group to demonstrate chaotic light scattering for his Junior Independent Study self-design project. He also used a computer simulation of the same experiment to help with some limitations in visualizations. He created a new arrangement of spheres and observed several of the same properties as the original group, including the Wada Property [3].

B. Fractals and Fractal Dimensions

The term fractal was coined by Benoît Mandelbrot in the winter of 1975 while preparing for his first publication. A name was needed for his new type of geometry, shapes, and dimensions. The word fractal comes from the verb "frangere" (to break). [4].

As James Gleick said,"In the mind's eye, a fractal is a way of seeing infinity" [4]. Fractal geometry is a way of dealing with complex systems that have no characteristic length scale. One well known example of this is a coastline. In Mandelbrot's 1967 paper, *How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension*, he examined self-similar curves that have non-integer dimensions between 1 and 2. Fractal dimensions were also created by Mandelbrot. They are a way of measuring qualities that otherwise have no clear definition, such as the degree of roughness, brokenness, or irregularity of an object. He determined specific ways of calculating the fractal dimension of real objects. This allowed his new geometry to describe the irregular patterns found in nature while the degree of irregularity remains a constant over different scales.

C. Self-Similarity

A self-similar object is almost identical to a part of itself, so the whole has the same shape as one or more of the parts. There are three types of self-similarity: exact, approximate, and statistical. Exact self-similarity only occurs in mathematically defined fractals where constraints from the physical world do not apply. This means the fractals are not naturally formed objects, but objects that can only exist by a strict mathematical definition. The most famous example of this is the Koch snowflake. Because it is both symmetrical and scale-invariant it can be magnified continuously without changing shape. Approximate self-similarity is the most common. It is seen when looking at an object at different scales because structures are seen that are recognizably similar but not identical. One example of this is the mathematically defined system called the Mandelbrot set. Approximate self-similarity can also be found in surprising places in nature like in the leaves of a fern. Almost all self-similar patterns are approximately self-similar. Sometimes the self-similarity isn't visually obvious but there may be numerical or statistical measures that are preserved across scales. One example of this is the fractal dimension of 1/f noise [4, 7].

II. THEORY

A. Chaotic Light Scattering

Before describing chaotic scattering, scattering itself must first be defined. Generally speaking, it is the problem of obtaining the relationship between an input variable characterizing an initial condition for a dynamical system and an output variable characterizing an appropriate defined final state of the system. One example of this is the motion of a point particle in a potential $V(x)$, where $V(x)$ is zero or very small. Outside the scattering region, the orbit moves in a straight line. The orbit approaching the scattering region interacts with the scatter then leaves the region [1]. In chaotic scattering, however, an initially freely moving orbit enters a scattering region and evolves chaotically before it escapes and returns again to free motion [2]. Thus, a small change in the direction of the approaching orbit could cause a great change in the exit direction.

B. Fractal Dimensions

A fractal is a fine structure at arbitrarily small scales. It is generally too irregular to be easily described in traditional Euclidean geometric language. Also, it is usually self-similar (at least approximately or statistically) and has a simple and recursive definition. [6]

In fractal geometry, the fractal dimension D is a statistical quantity that gives an indication of how completely a fractal appears to fill space, as one zooms down to finer and finer scales. For example, a point has a dimension of 0, a line has a dimension of 1, and a surface has a dimension of 2. But a fractal like Koch's snowflake, a coastline, or a leaf will have a fractal dimension between 1 and 2. Going into the 3rd dimension then deals with volume. In between 2 and 3 is a crumpled 2D object, like aluminum foil, cauliflower, a brain, or a lung [7].

To get the dimension of a fractal, first suppose we have a number of boxes with all the same side length r. The boxes have area r^n. Cover the fractal in a $N(r)$ number of boxes. The boxes are then reduced in size by the scaling factor of $(1/r)^n$, and the number $N(r)$ is determined again. We can calculate $N(r)$ as follows:

$$s^2 = N(r) * r^n. \tag{1}$$

By moving the area r^n to the other side, we get

$$N(r) = \frac{s^2}{r^n}. \tag{2}$$

Because s^2 is a constant, we can replace it with C. This gives

$$N(r) = C\Big(\frac{1}{r}\Big)^n. \tag{3}$$

We must solve for n and then take the limit as r goes to zero to get the dimension. Doing this yields

$$n = \frac{\ln N(r) - \ln C}{\ln(1/r)} \tag{4}$$

We can ignore the constant because it is negligible. We finally take the limit of the equation above

$$D = \lim_{r\to 0} \frac{\ln N(r)}{\ln(1/r)}, \tag{5}$$

which gives us the fractal dimension of the object [4, 5, 8, 9].

C. Self-Similarity

A self-similar object is exactly or approximately similar to a part of itself. A set C is called self-similar if it is the union of small copies of itself. Self-similar structures are scale invariant. When zooming into a picture of the image, the image does not change.

A function is defined as homeomorphic only if it is continuous, one-to-one, and onto. This means that the limit of the function must exist everywhere in its domain, each x value of the function corresponds to only one y value, and for every y in the domain there is an x in the domain such that $f(x) = y$. The topological definition states that a closed and bounded topological space X containing $\mathbb{R}^n$ for some natural number n is self-similar if there exists a finite set S indexing a set of homeomorphisms $\{f_s\}_{s\epsilon S}$ for which

$$X = \bigcup_{s\epsilon S} f_s(X). \tag{6}$$

Eq. 6 states that X is a union of a number of small copies of itself.

Self-similar objects with parameters N and s are described by as power law such as $N = s^d$, where

$$d = \frac{\ln N}{\ln s} \tag{7}$$

is the dimension of the scaling law [9].

III. PROCEDURE

A. Setup

The experiment used four 10-inch mirrored spheres stacked in an upside down pyramid. Three of the four sides were covered in colored cloth. After testing three different colors were chosen: red, blue, and light green.

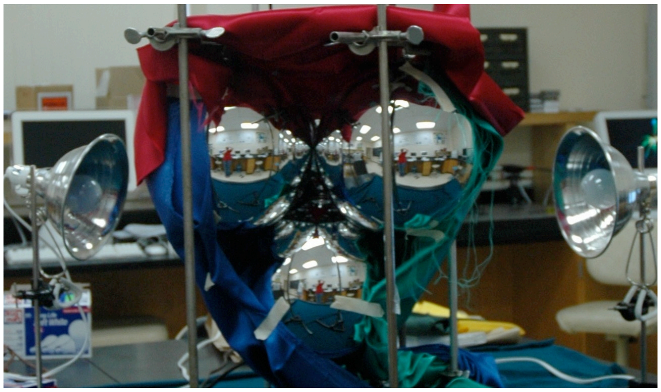

FIG. 1: The setup of the experiment. Note the three different color cloths, the three different lamps (the third is above the red cloth and outside the picture), and the uncovered side. This side was where all photographs where taken.

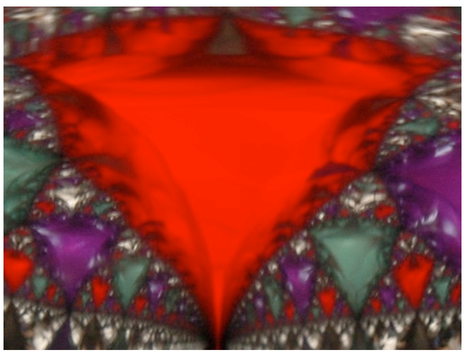

FIG. 2: The center of the fractal pattern. The fractal dimension of this shape was determined.

These three cloths were taped to the outside by masking tape. The fourth side was the side used to take pictures. Outside of the cloths, three lights were set up. Fig 1 shows the final version of the setup.

B. Using Photographs to Look at Fractal Dimension and Self-Similarity

The main part of this experiment included analyzing photographs. The pictures were taken with a Nikon D70 digital camera. With the guidance of the camera manual, various setting were tried, including adjusting the aperture and the exposure, but nothing seemed to work. With some outside help, the pictures came out well enough that they could be manipulated with CANVAS and GRAPHIC CONVERTER.

The photos were all taken in the dark either with the camera on the tripod or while sitting on the table with the camera being held as close as possible. There were problems with the pictures being too fuzzy, but by taking them with the RAW setting on, so the the images were not compressed, the fuzziness around the edges decreased.

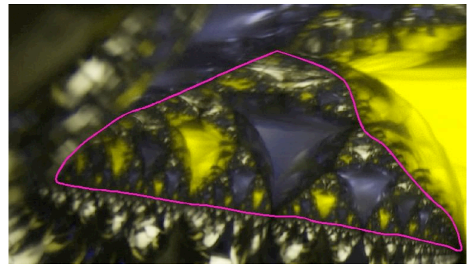

FIG. 3: The triangle is outlined in pink. This one of the three side triangles of the fractal pattern. The fractal dimension of this was determined. By symmetry, its fractal dimension is equal to the other two.

1. Fractal Dimensions

Fractal dimensions for four areas were determined. Only two of these areas needed to be calculated due to symmetry in the image. The four areas were the center and the large triangles along the sides of the center. They can be seen in Figs 2 and 3 respectively.

The photographs had to be worked with extensively, as they were quite blurry. Note that the colors in Fig. 3 is different from Fig. 2 and the other pictures. In order to see the top edge of the side triangle, the colors were changed using the Deutan Color Blindness effect in GRAPHIC CONVERTER and the picture was enlarged. For the picture shown in Fig. 2, only enlarging was necessary, so no color change was done. A border was drawn around the actual fractal with a pencil to tell exactly where the fractal was.

Calculating the fractal dimension was done by estimating the perimeter by using a divider compass. An initial width of the step size was chosen and the number of steps it took to go around the fractal was counted. This was repeated four times by varying the width of the compass. The width of the compass and the perimeter were plotted on a log-log plot and the data was fit to a power law $p = cs^{1-D}$, where D is the dimension. The dimension D can then be solved for.

2. Self-Similarity

For self-similarity, the photographs required less manipulation. There were still some problems with clarity, which limited the extent of analysis possible. To analyze the photos, CANVAS was used to enlarge the images and outline diamonds. The diamond was resized while the proportions were kept the same. CANVAS calculated the area of the diamond and the length of the left most side. The length was plotted versus the area to find a correlation between the two. If the slope of the line is equal to 2, then self-similarity is present. The relation of the length of the side to the area of the diamond is $A = \alpha L^2$. Taking the log of both sides, gives $\log A = \log \alpha + 2 \log L$ The slope is the value in front of $\log L$.

IV. RESULTS AND ANALYSIS

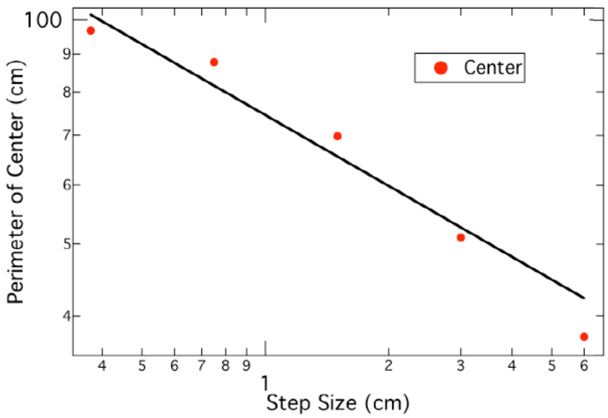

FIG. 4: The center red diamond of the image has a fractal dimension of 1.32 ± 0.04. This was found by plotting the step size in centimeters versus the perimeter of the object in centimeters on a log-log scale. The data were then fit to a power law. The power is $1 - D$, where D is the fractal dimension.

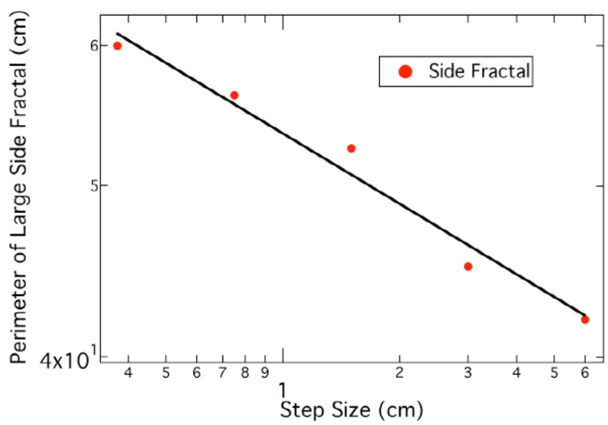

FIG. 5: The side fractals of the image have a fractal dimension of 1.13 ± 0.06. This was only determined for one of them, but due to the symmetric nature of the image, it is assumed for the other two.

A. Fractal Dimensions

Only two fractals that were analyzed for dimension but by symmetry, four were actually seen in the image. The first fractal was the large red shape in the middle of the image. The dimension of this is 1.32 ± 0.04. This means that it has more dimension than a line, but less than a surface. It has a larger dimension than Koch's snowflake by about 0.06. Fig. 4 shows a plot of the data and the fit used to determine the dimension.

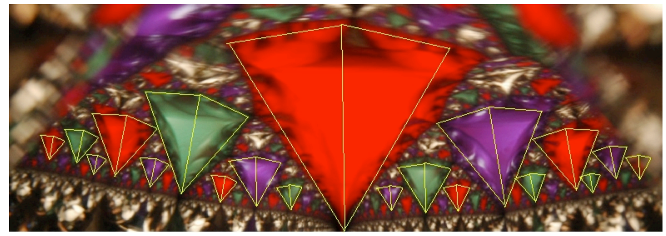

FIG. 6: The green and purple diamonds are symmetrically equivalent. The red diamonds are different sizes than the green and purple.

The side fractals were more difficult to analyze because of the curved surface of the spheres. It was difficult to get a picture in which the entire image was in focus. The fractal dimension of these three are 1.13 ± 0.06. The dimension is less than the dimension of the center.

B. Self-Similarity

The self-similarity was first broken up into two parts. Note that in Fig. 6, the green and purple are the same due to the symmetry across the middle. The analysis of the self-similarity was broken up into two parts because it was thought that the the red diamonds might be filling in sizes for where the green and purple do not have triangles of that size.

First the red was examined. Self-similarity was found because the data had a slope of 2.048 ± 0.001. Fig 6 shows the red diamond used; they are outlined in yellow.

Next the green and purple areas were analyzed. The self-similarity was also found because the slope of the line was 1.954 ± 0.003. This includes the diamonds on either side of the symmetric split. The green and purple diamonds are shown in Fig. 6. They are all outline in yellow.

By taking all of the data and plotting it on one graph we can check to see if there is any correlation between all of the diamonds; we see that there is. The self-similarity present with all of the diamonds because the slope is 2.046 ± 0.004. Fig 7 shows the plot of all the data and the power law fit used to check for self-similarity. It shows the area of the diamond versus the length of the left side on a log-log plot. A power law fit was done to determine if self-similarity was present. Looking at the plot, we see that the red diamonds are sizes in between the green and purple diamonds. However, there is not enough data to determine a pattern; the red diamonds became too small to quickly and the photograph could not be magnified any more without the images becoming indistinguishable. As shown in Fig. 7, there appears to be a pattern starting in the upper right corner and moving down the line towards the left. Moving down the line, there is one green/purple and one red again. The next group changes as there are three green and purple diamonds and then two red ones. Lastly we have three more green and purple diamonds.

This is where the red diamonds became too small to determine where the edges were. We do know that they are at least smaller than the lowest point on the plot.

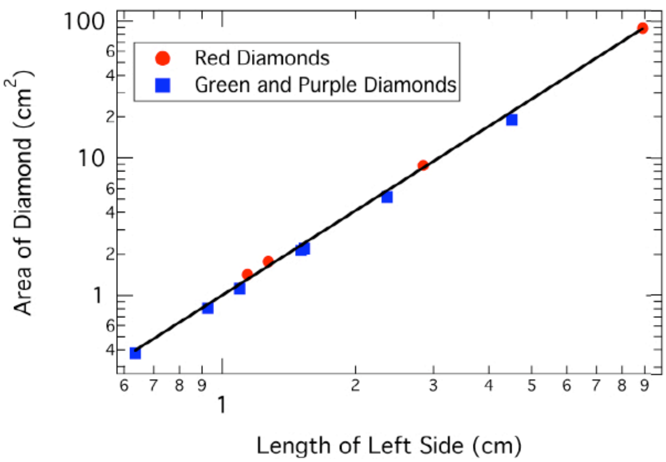

FIG. 7: By combining all the data onto one plot we can tell that all the diamonds are self-similar. Interestingly, the red diamonds are sizes in between the sizes of the green and purple. However, no conclusions can be drawn from this statement.

V. CONCLUSION

The fractal dimension of the red diamond at the center of the image is 1.32 ± 0.04. This value is just greater than Koch's snowflake, indicating that the center fractal is more than a line but less than a surface. The fractal dimension of the three outer, large fractals is 1.13 ± 0.06. This dimension is significantly less than the center. It is still more than a line and therefore greater than dimension one. One problem with these values, however, is that they might not be the best representation. Because the light is reflecting off a curved surface, the borders were curved at places and fuzzy. These values are correct for the calculations possible and the methods used, but there may be a more accurate method.

The self-similarity found in the image by looking at the length versus the area of diamonds an section of the image. The slope of the data was calculated to be 2.046 ± 0.004. To be self-similar, this value must be 2 and therefore, the fractal is self-similar. However, there were not enough data to determine whether the size of the red diamonds have any specific relationship to the sizes of the green and purple diamonds, although there does appear to be a possible pattern to the size. There were some problems with the accuracy due to the shape of the surface. Also, due to the quality of the photographs, the scope of the analysis was limited.

Possible areas of future work include changing the setup of the spheres and analyzing the fractal dimensions and existence of self-similarity in different areas of the image formed.

VI. ACKNOWLEDGMENTS

Thanks go to Hannah Samuell for coming in and taking an hour's time out of working on her own junior IS to help me learn how to use a camera propertly. Secondly, thanks to Dr. Ramsay for many humorous explanations of the topology. Lastly, thanks to Dr. Jacobs for his extensive help and assistance with this experiment. He was more help than I could have imagined.

[1] E. Ott and T. Tél, *Chaos* **3** 417-426 (1993).
[2] D. Sweet, E. Ott, and J. A. Yorke, *Nature* **399**, 315-316 (1999).
[3] D. J. Miller, Jr. IS Self-Design thesis, The College of Wooster, (2001).
[4] J. Gleick, *Chaos: Making a New Science* (Viking Penguin Inc., New York, 1987).
[5] *Wikipedia*, www.wikipedia.org (2008).
[6] K. J. Falconer, *Techniques in Fractal Geometry* (John Willey and Sons, New York, 1997).
[7] A. Bunde and S. Havlin, *Fractals in Science* (Springer-Verlag, Berlin, 1994).
[8] G. Elert, http://hypertextbook.com/chaos/ (2007).
[9] E. W. Weisstein, http://www.mathworld.wolfram.com (2008).
[10] K. Alligood, T. D. Sauer, and J. A. Yorke, *Chaos - An Introduction to Dynamical Systems* (Springer, New York, 1997)
[11] S. Bleher, E. Ott, and C. Grebogi, *Phys. Rev. Lett.* **63**, 919-922 (1989).
[12] Q. Chen, M. Ding, and E. Ott, *Phys. Lett. A* **115**, 93-100 (1990).