Current basins of attraction in inertia ratchets

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Abstract

Inertia ratchets are characterized by a complex dynamics with multiple current reversals. We recently studied the problem of control of current on inertia ratchets and demonstrated the feasibility of a control method associated with a process of locking to different mean velocity attractors. Here we present the results of a study of the fractal characteristics of basins of attraction for different mean velocities as well as the influence of quenched disorder on the attractors. We find that our previous conjecture that the domains of attraction of the two velocities are intermixed fractals is verified. We also find that the attractor for the negative velocity dissolves as the strength of disorder is increased. These results have important implications for the design of a reliable technique for controlling the particle mean velocities.

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1. Introduction

Thermal ratchets \cite{2} are simple stochastic models where a non-zero net drift speed may be obtained from time correlated fluctuations interacting with asymmetric periodic structures. Initially, ratchets received attention as simple models of motor proteins \cite{3}. Biological transport processes such as the operation of molecular motors are well approximated by overdamped dynamics. Inertial ratchets were introduced later \cite{4} as models for nanoscale friction \cite{5}, sorting and separation of microscopic and mesoscopic particles \cite{6}, surface smoothening \cite{7}, and micron-scale devices \cite{8}. Inertial ratchets,
even in the absence of noise, are characterized by a more complex dynamics than overdamped ratchets including chaotic motion and multiple reversals in the current direction [4,9]. It has been suggested [9] that current reversal may be related to a bifurcation in which a chaotic ratchet state suddenly becomes periodic. It was shown later [10] that current reversals can occur even in the absence of bifurcations from chaotic to periodic motion when associated with phase locking phenomena. Thus, it would be of interest to propose a reliable method for the control of multiple current reversals in inertial ratchets.

The problem of controlling the current reversal has been addressed in a recent publication by Barbi and Salerno [11] who studied the influence of weak periodic signals on the transport properties of underdamped ratchets. They found that the constant current intervals related to the ratchet can be significantly enlarged by a weak subharmonic signal that is in phase with the internal driver. This stabilization phenomenon is found to exist both in the absence and in the presence of temporal noise.

In this paper, we study the fractal characteristics of the basins of attraction for different velocities with and without quenched disorder. In a previous paper [1], we proposed a control method for current reversals associated with a process of locking to different mean velocity attractors. This control mechanism requires tracking the trajectory of the moving particles in order to find the precise position and velocity of the particles in different basins of attraction for a specific velocity. The characteristics of the basins of attraction corresponding to different velocities are very important for the reliability of the control method.

Another important issue for the actual realization of current control is the influence of disorder. We recently [12] observed that in the presence of quenched disorder, diffusion in overdamped rocking ratchets can eventually reach the same order of magnitude as regular drift velocity. On the other hand, the effect of quenched disorder on an underdamped rocking ratchet [13] is that current reversal and chaotic diffusion may take place on otherwise regular trajectories and some chaotic trajectories become regular.

2. Model

In scaled non-dimensional coordinates the equation of motion is given by [13]

\[ \varepsilon \ddot{x} + \gamma \dot{x} = \cos(x) + \mu \cos(2x) + \Gamma \sin(\omega t) + \alpha \zeta(x), \]  

(1)

where \( \varepsilon \) is the mass of the particle, \( \gamma \) is the damping coefficient, \( \Gamma \) and \( \omega \) are, respectively, the amplitude and frequency of an external oscillatory forcing. The unperturbed ratchet potential, with period \( \lambda = 2\pi \), is given by

\[ U(x) = -\sin(x) - \frac{\mu}{2} \sin(2x) \]  

(2)

and has been the subject of extensive studies mainly in models with no disorder [4,9,10] and with disorder [12,13,1].

The addition of a quenched disorder term \( \alpha \zeta(x) \) gives a more realistic representation of the substrate. The coefficient \( \alpha \geq 0 \) is the strength of this quenched disorder and \( \zeta(x) \)
are independent, uniformly distributed random variables with no spatial correlations, corresponding to a piecewise constant force on the period of the potential.

Let us briefly review some relevant results already reported in Ref. [1] for the model without quenched noise ($\zeta=0$). Two experimentally inspired methods may be used to study the dynamics of both single and multiparticle systems as the amplitude of the external oscillatory forcing is increased. In the first method, the same initial condition is used for each new value of the amplitude $\Gamma$. In the second method that is history dependent, the last position and velocity of the particle evolving under an amplitude $\Gamma$ are the new initial conditions when the amplitude $\Gamma$ is changed.

Figs. 1a, c and e show the normalized mean velocity $\langle v \rangle/v_o$ (with $v_o = \lambda/T$ and $T = 2\pi/\omega$) as a function of the external force strength $\Gamma$, using both methods. The corresponding bifurcation diagrams are shown in Figs. 1b, d and f. In this paper, we are especially interested in the region with $\Gamma \in [0.83; 1.05]$. Fig. 1a reveals that with method I, the normalized mean velocity has only two values: $+1$ and $-1$. A number of transitions take place between the two mean velocities as $\Gamma$ varies showing that the system is extremely sensitive to the value of $\Gamma$. On the other hand, as can be seen in Figs. 1c and e, while the possible values for the normalized mean velocity for the same range of values of $\Gamma$ are again $+1$ or $-1$ in the case of method II, the particle remains locked in one of the values as $\Gamma$ is changed. The locking value depends on the initial value of $\Gamma$. For example, Fig. 1c corresponds to a starting value of $\Gamma = 0.89665$ and here the particle remains locked to a velocity of $+1$, while Fig. 1e shows that with an initial value of $\Gamma = 0.89666$ the mean velocity of the particle is locked to $-1$.

3. Results

We recently [1] conjectured that the two mean velocity attractors, corresponding, respectively, to mean velocities $+v_o$ and $-v_o$, coexist with a fractal boundary between their basins of attraction. Our goal here is to verify this conjecture and to elucidate the structure of the basins of attraction. We are also interested in understanding the effect that quenched (spatial) disorder can have on the structure of the domains of attraction, because this will enable us to develop effective control strategies using quenched disorder as the control parameter.

To verify our conjecture [1] we begin with the case of $\zeta=0$, i.e., no quenched disorder. The numerical solution of Eq. (1) is obtained with a variable step Runge–Kutta–Fehlberg method [14]. To make contact with our previous work [1], we let $\varepsilon = 1.1009$, $\gamma = 0.1109$, $\mu = 0.5$, $\Gamma_o = 0.89665$ and $\omega = 0.67$. The initial conditions are selected from a grid of $512 \times 512$ points in the rectangular region limited by $x_0 \in [-\pi, \pi]$ and $v_0 \in [0, 2]$. These ranges are chosen to cover a whole spatial period for the potential in the position $x$, and the velocities in order that kinetic energies extend from 0 up to the height of the potential barrier. For each initial condition the trajectory is discarded during a transitory time taken as $400T$ and then the permanent regime is studied during $100T$.

In the region $\Gamma \in [0.83, 1.05]$ only two types of solutions exist as discussed above. In order to elucidate this point, we denote with a solid circle an initial condition
$(x_0, \nu_0)$ that leads to a trajectory with a normalized mean velocity $-\nu_0$, and we denote with an open circle an initial condition that leads to a trajectory with a normalized mean velocity $+\nu_0$. In this way, the basins of attraction for both mean velocity attractors were obtained, as shown in Fig. 2a–d. Fig. 2b–d are successive enlargements of Fig. 2a and the evidence of a typical fractal behavior emerges from them. The solid lines in these figures are equipotential curves with the corresponding potential value printed on them. We used a box counting method to study the fractal nature of the mean velocity domains of attraction. Our results show that the mean velocity attractors are fractal with a dimension $d = 1.87$ for the negative mean velocity domain of attraction.

We carried out extensive numerical study with finite $\nu, 1 \geq \nu \geq 0$ in order to determine the effects of quenched noise on the structure of the domains of attraction. Typical results with increasing $\nu$ are shown in Fig. 3a–c. Initial conditions are found from a grid of $256 \times 256$ points. We find that while increasing the quenched noise level does not modify the global shape of the domains, the number of points in the negative mean velocity attractor diminishes as $\nu$ is increased indicating that the negative domain of attraction disappears with increasing $\nu$. In order to verify that the shape of the velocity domain remains unchanged, Fig. 3d shows the results for the same strength of disorder as Fig. 3b, but with initial conditions from a larger grid of $1024 \times 1024$ points. We note that all figures show that the negative attractor dissolves as $\nu$ increases. This result is summarized in Fig. 4 where the function

$$P(\nu) = \frac{N_-}{N_- + N_+}$$

with $N_+(N_-)$ number of initial conditions corresponding to $+(-)1$ velocity domain, is plotted against $\nu$. This figure shows how the number of initial conditions in the domain of the negative mean velocity attractor dramatically diminishes for $\nu \approx 0.0075$.

The above results were obtained by studying the trajectory of only one particle, but ratchet transport is essentially stochastic. We use a collection of particles with different initial conditions to study the transport phenomena. We work with an ensemble consisting of hundred particles having identical initial velocities $v_0$, but with initial positions equally distributed in the range $[x_{\text{min}}, x_{\text{max}}]$. The initial probability density is
Fig. 2. (a) Basins of attraction of $-v_0$ (black dots) and $+v_0$ without quenched disorder ($x=0$). $L'=0.89665$. The initial conditions are selected on a grid of $512 \times 512$ points. The solid lines are equipotential. (b)–(d) are successive enlargements of (a) to show the fractal nature of the basins of attraction.

given by

$$\rho(x,v,0) = \delta(v - v_0)[H(x - x_{\text{min}}) - H(x - x_{\text{max}})].$$  (4)

The particles move in an inhomogeneous media with two regions separated by an interface located at $x=0$. Some representative results are shown in Fig. 5 where the stroboscopic normalized position of each particle is plotted as a function of the normalized time. Fig. 5a shows the case with small quenched noise. The particles separate into two groups moving with positive and negative mean velocities, respectively. The group moving to the left enters the disordered medium and continues moving without any change. As $x$ increases, more and more particles that were initially going to the left get dispersed and reflected to the right. This diminishes the negative current (Fig. 5b and c). This result implies that $x$ may be used as a control parameter for the positive and negative currents. For $x \cong 0.008$ all particles are finally moving to the right. The depth of penetration decreases as $x$ increases beyond this threshold. Whenever a disorder threshold that depends on the mass of the particle is reached, the localization effect [13] sets in. The localization may be seen in Fig. 5d where a high-disorder strength leads to the localization of particles near 0.
Fig. 3. Basins of attraction of $-\nu_0$ (black dots) and $+\nu_0$ with quenched disorder with $\Gamma = 0.89665$. The initial conditions of (a)–(c) are selected on a grid of $256 \times 256$ points. The initial conditions of (d) are selected on a grid of $1024 \times 1024$ points. (a) $\alpha = 0$, (b) $\alpha = 0.005$, (c) and (d) $\alpha = 0.008$.

Fig. 4. $P(\alpha)$ as a function of the quenched noise strength for $\Gamma = 0.89665$. 
Fig. 5. Stroboscopic positions of an ensemble of 100 particles moving in an inhomogeneous media. All the particles start with velocity $v_0 = 1$ and initial positions between $x_{\text{min}} = 50\lambda$ and $x_{\text{max}} = 51\lambda$. The region $x > 0$ has no spatial disorder. The region $x < 0$ has quenched disorder with strength $\alpha$. (a) $\alpha = 0.001$, (b) $\alpha = 0.005$, (c) $\alpha = 0.008$, and (d) $\alpha = 0.6$.

4. Conclusions

We have studied the domain of attractions of two different mean velocities that characterize the inertial ratchet of Eq. (1). We find that our conjecture [1] that the two attractors are intermixed fractals is verified. In addition, our results show that quenched disorder has significant effect on the basins of attraction. In particular, with increasing disorder the direction of particles can be reversed, leading to disappearance or weakening of the negative velocity attractor. These results have important implications since they can be utilized in the design of reliable techniques for controlling particle mean velocities.

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